

Dyadic C^2 Hermite interpolation on a square mesh

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Abstract

For prescribed values of a function and its partial derivatives of orders 1 and 2 at the vertices of a square, we fit an interpolating surface. We investigate two families of solutions provided by two Hermite subdivision schemes, denoted HD^2 and HR^2 . Both schemes depend on 2 matrix parameters, a square matrix of order 2 and a square matrix of order 3. We exhibit the masks of both schemes. We compute the Sobolev smoothness exponent of the general solution of the Hermite problem for the most interesting schemes HD^2 and HR^2 and we get a lower bound for the Hölder smoothness exponent. We generate a C^2 interpolant on any semiregular rectangular mesh with Hermite data of degree 2.

Key words: Hermite interpolation; subdivision; surfaces; Hölder condition

1 Introduction

Before defining Hermite subdivision schemes, we describe some notations. \mathbb{N}_0 refers to the set of nonnegative integers. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $|\mu| := |\mu_1| + \dots + |\mu_s|$, $\mu! := \mu_1! \cdots \mu_s!$ and $\partial^\mu := \partial_1^{\mu_1} \cdots \partial_s^{\mu_s}$, where ∂_j denotes the partial differentiation operator with respect to the j th coordinate. Denote

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$M = M(r, s) := \{\mu \in \mathbb{N}_0^s : |\mu| \leq r\}$ and by $\#M$ we denote the cardinality of the set M . Now the elements in M can be ordered in such a way that $\nu = (\nu_1, \dots, \nu_s)$ is less than $\mu = (\mu_1, \dots, \mu_s)$ if either $|\nu| < |\mu|$ or $|\nu| = |\mu|$, $\nu_j = \mu_j$ for $j = 1, \dots, i-1$ and $\mu_i < \nu_i$ for some $1 \leq i \leq s$.

A *multivariate Hermite subdivision scheme of degree r* , \mathcal{H} , is a recursive scheme for computing a function $\phi : \mathbb{R}^s \rightarrow \mathbb{R}$ and its partial derivatives of order less or equal to r . The initial data of the Hermite subdivision scheme is a function $f_0 : \mathbb{Z}^s \rightarrow \mathbb{R}^M$. For every $\alpha \in \mathbb{Z}^s$, $f_0(\alpha)$ is a vector whose components are $f_0^\mu(\alpha)$ for $\mu \in M$.

The component $f_0^{(0,0,\dots,0)}$ is a control value for ϕ , the s components $f_0^{(1,0,\dots,0)}$, $f_0^{(0,1,\dots,0)}$, \dots , $f_0^{(0,0,\dots,1)}$ are the control values for the partial derivatives of order 1, $\phi_1, \phi_2, \dots, \phi_s$, the other components are the control values for the partial derivatives of higher orders. The sequence of refinements $\{f_n : \mathbb{Z}^s \rightarrow \mathbb{R}^M\}_{n>0}$ is recursively defined through a matrix function $A : \mathbb{Z}^s \rightarrow \mathbb{R}^{M \times M}$ with finite support by

$$D^{n+1}f_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta)D^n f_n(\beta), \quad \alpha \in \mathbb{Z}^s, n \in \mathbb{N}_0, \quad (1)$$

where $D \in \mathbb{R}^{M \times M}$ is the diagonal matrix whose diagonal element of index μ is $1/2^{|\mu|}$.

The matrix function A is called the *mask of type (r, s)* of \mathcal{H} . The *support* of the mask is $\{\alpha \in \mathbb{Z}^s : A(\alpha) \neq 0\}$. One should consider that the control vector $f_n(\alpha)$ is attached to the dyadic point $\alpha/2^n$ of \mathbb{R}^s .

The explanation of the above formula is not immediately obvious, it is related to the rule of derivation for $\partial^\mu \phi(\cdot/2^n)$ and it is deduced after considering various examples. A Hermite subdivision scheme is *interpolatory* if $A(0) = D$ and $(\forall \alpha \in \mathbb{Z}^s, \alpha \neq 0) A(2\alpha) = 0$. In this paper, only interpolatory schemes will be considered.

For the dimension $s = 1$, the first investigations of Hermite-type subdivision schemes of degree larger than 1 have been made by Merrien [8], Dyn and Levin [2], Zhou [12], Han [4] and Yu [10]. They have given the theory and the tools for analyzing the convergence and smoothness of Hermite-type interpolatory schemes.

In this paper, we exhibit bivariate Hermite subdivision schemes of degree 2 which are C^2 and whose support is the unit square. We suppose that we know the values of a function f and of its partial derivatives of orders 1 and 2 at the vertices of the unit square $[0, 1]^2$. These values are known as Hermite data. We are looking for a function f defined on the unit square interpolating these Hermite data. Two families of solutions will be provided

by two Hermite subdivision schemes, denoted HD^2 and HR^2 . Both schemes depend on 2 matrix parameters, a square matrix Λ of order 2 and a square matrix K of order 3. They can also be applied to a rectangular net of points in \mathbb{R}^2 .

Each algorithm is local and the construction on a square is independent of its neighbors. In order to get C^2 continuity across an edge, the construction depends only on the length of this edge and on the values of f and its derivatives at the endpoints of the same edge. Therefore, HD^2 and HR^2 provide algorithms building C^2 Hermite interpolants with data on a square mesh.

This paper is organized as follows. In Section 2, we recall two univariate Hermite subdivision schemes HC^1 and HC^2 as given by Merrien [7,8]: HC^1 is defined by Λ , HC^2 is defined by K . In Section 3, we introduce both recursive schemes HD^2 and HR^2 for interpolation in the plane. The scheme HD^2 is constructed by using HC^1 and HC^2 along the edges and the diagonals of the unit square. The scheme HR^2 is constructed by using HC^1 and HC^2 along the edges of the unit square and the two medians $x = 1/2$ and $y = 1/2$. We exhibit the masks of both schemes HD^2 and HR^2 .

In Section 4, we recall the definition of convergence of a Hermite interpolatory subdivision scheme and the definition of the basic matrix function $\Phi : \mathbb{R}^s \rightarrow \mathbb{R}^{m \times m}$ of a C^r scheme. The first row ϕ of Φ provides a solution to a refinement equation and the smoothness of the solution of the original Hermite problem comes from the smoothness of ϕ . In Section 5, we recall a convergence criterion for Hermite subdivision schemes defined by Han [5]. With this criterion, we will be able to compute the Sobolev smoothness exponent of ϕ . In Section 6, we apply the results of the previous section to the schemes HD^2 and HR^2 . If the two matrices Λ and K are appropriately chosen for each scheme, the interpolation proposed by HD^2 is at least $C^{2.83}$ and the interpolation proposed by HR^2 is at least $C^{2.91}$. The mathematical justification of these results are given in the Appendix. In Section 7, we describe the construction of a C^2 interpolant on a semiregular rectangular mesh with Hermite data of degree 2. We conclude with some open questions.

2 Two univariate algorithms

We recall two univariate versions of the subdivision scheme for Hermite interpolation, given by Merrien [7,8]. Firstly we define HC^1 . Suppose that we know the values of a function f and of its first derivative $p = f'$ at the endpoints of a bounded interval I of \mathbb{R} . To build f and p , we proceed by induction on $n \in \mathbb{N}_0$ and we assume that f, p are already known on the regular partition \mathcal{P}_n of I in 2^n subintervals. Let \mathcal{P}_{n+1} be the regular partition of I in 2^{n+1} subintervals.

If $x \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$, then we compute f and p at x according to the following scheme, which depends on a matrix parameter $\Lambda = \begin{pmatrix} \lambda_{00} & \lambda_{01} \\ \lambda_{10} & \lambda_{11} \end{pmatrix}$:

$$\left. \begin{aligned} f(x) &= \lambda_{00}[f(a) + f(b)] + \lambda_{01}h[p(b) - p(a)] \\ hp(x) &= \lambda_{10}[f(b) - f(a)] + \lambda_{11}h[p(a) + p(b)] \end{aligned} \right\}, \quad (2)$$

where $h = |I|/2^n$, $a = x - h/2$, $b = x + h/2$.

By applying these formulae on ever finer partitions, we define f and p on a dense subset of I . In order to get that f and p are uniformly continuous on I and can be extended on I , some conditions should be satisfied by Λ . The first necessary ones are that $\lambda_{00} = 1/2$ and $\lambda_{10} + 2\lambda_{11} = 1$ (see Proposition 1 of [7]). These two conditions will always be assumed.

The scheme is C^1 -convergent if f and p can be extended on I into continuous functions with $p = f'$. As in [9], when $I = [0, 1]$, for $\alpha = 0, \dots, 2^n$,

let $U_n^\alpha = \begin{pmatrix} p(\alpha/2^n + h) - p(\alpha/2^n) \\ [f(\alpha/2^n + h) - f(\alpha/2^n)]/h - [p(\alpha/2^n + h) + p(\alpha/2^n)]/2 \end{pmatrix}$ and for

$\epsilon = \pm 1$, let $A_\epsilon = \begin{pmatrix} 1/2 & \epsilon\lambda_{10} \\ \epsilon(1/4 + \lambda_{01}/2) & 1/2 + \lambda_{11} \end{pmatrix}$. Then $U_{n+1}^{2\alpha} = A_1 U_n^\alpha$ and

$U_{n+1}^{2\alpha+1} = A_{-1} U_n^\alpha$. The scheme is C^1 -convergent if and only if the generalized spectral radius $\rho(\{A_1, A_{-1}\}) < 1$. An equivalent condition is that there exists a matrix norm $\|\cdot\|$ such that: $\|A_\epsilon\| < 1, \epsilon = \pm 1$. In [9] we added that if $\rho = \rho(\{A_1, A_{-1}\}) < 1$, then the function $f' = p$ is Hölder continuous with exponent $-\log_2(\rho)$. Notice two important choices of Λ :

- (1) if $\Lambda = \begin{pmatrix} 1/2 & -1/8 \\ 3/2 & -1/4 \end{pmatrix}$, then f is the Hermite cubic interpolant,
- (2) if $\Lambda = \begin{pmatrix} 1/2 & -1/8 \\ 2 & -1/2 \end{pmatrix}$, then f is the Hermite quadratic spline interpolant with one knot at the midpoint of I .

Now, similarly to define HC^2 , suppose that we know the values of a function f and of its first and second derivatives $p = f', r = f''$ at the endpoints of an interval I of \mathbb{R} . With a similar algorithm, we recursively define f , p and r on \mathcal{P}_{n+1} according to the following scheme, which depends on a matrix parameter

$K = (\kappa_{k\ell}), k, \ell = 0, 1, 2$

$$\left. \begin{aligned} f(x) &= \kappa_{00}[f(b) + f(a)] + \kappa_{01}h[p(b) - p(a)] + \kappa_{02}h^2[r(b) + r(a)] \\ hp(x) &= \kappa_{10}[f(b) - f(a)] + \kappa_{11}h[p(b) + p(a)] + \kappa_{12}h^2[r(b) - r(a)] \\ h^2r(x) &= \kappa_{20}[f(b) + f(a)] + \kappa_{21}h[p(b) - p(a)] + \kappa_{22}h^2[r(b) + r(a)] \end{aligned} \right\}, \quad (3)$$

where $h = |I|/2^n$, $a = x - h/2$, $b = x + h/2$. The C^2 -convergence is similar to the C^1 -convergence. In order to get C^2 -convergence, some necessary conditions on K should be satisfied: $\kappa_{00} = 1/2$, $\kappa_{01} + 2\kappa_{02} = -1/8$, $\kappa_{10} + 2\kappa_{11} = 1$, $\kappa_{20} = 0$, $\kappa_{21} + 2\kappa_{22} = 1$ (see Proposition 3 of [8]). These conditions will always be assumed. General conditions for C^2 -convergence and many interesting choices of K can be found in [8].

The mask of the scheme HC^1 of matrix parameter Λ is defined as $A(-1) = D\Lambda$, $A(0) = D$ and $A(1) = D\Lambda'$, where $\Lambda' = ((-1)^{k+\ell}\lambda_{k\ell})$, and $A(\alpha) = 0$ if $|\alpha| > 1$. Similarly the mask of the scheme HC^2 with matrix parameter K is defined as $A(-1) = DK$, $A(0) = D$ and $A(1) = DK'$, where $K' = ((-1)^{k+\ell}\kappa_{k\ell})$, and $A(\alpha) = 0$ if $|\alpha| > 1$.

3 Two examples of recursive interpolation schemes on a square

We will describe two families of recursive interpolation schemes on the unit square $R = I \times I$. Our purpose is to define a bivariate function f on R . We expect that this function will be C^2 with partial derivatives: $p = f_x$, $q = f_y$, $r = f_{x^2}$, $s = f_{xy}$, $t = f_{y^2}$. At the beginning of the construction, the only data that are known about these six functions are their values at the vertices of R . For both families of schemes, the construction depends on 2 matrix parameters Λ and K of respective orders 2 and 3 defined in the previous section. The first family will do the computations by using the edges and the diagonals of the square R and of subsequent subsquares. The second family will do the computations by using the sides, the horizontal medians and the vertical medians of the square R and of subsequent subsquares.

3.1 The recursive scheme HD^2

For $n = 0, 1, 2, \dots$, let us denote again by \mathcal{P}_n the regular partition of I in 2^n subintervals. We proceed by induction on n and we assume that f, p, q, r, s, t are already known on the mesh $\mathcal{P}_n \times \mathcal{P}_n$. We will then define these functions on $(\mathcal{P}_{n+1} \times \mathcal{P}_{n+1}) \setminus (\mathcal{P}_n \times \mathcal{P}_n)$. Let $h = |I|/2^n$. (see Figure 1).

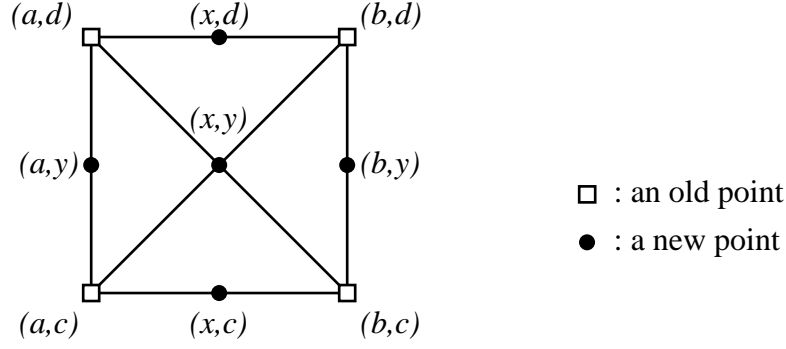


Fig. 1. Dyadic points at steps n and $n + 1$.

Case 1: $(x, y) \in (\mathcal{P}_{n+1} \setminus \mathcal{P}_n) \times \mathcal{P}_n$.

We set $a = x - h/2, b = x + h/2, c = y$. At first we use the scheme HC^2 defined in (3) to define $f, p = f_x, r = f_{x^2}$:

$$\left. \begin{aligned} f(x, c) &= \kappa_{00}[f(b, c) + f(a, c)] + \kappa_{01}h[p(b, c) - p(a, c)] + \kappa_{02}h^2[r(b, c) + r(a, c)] \\ hp(x, c) &= \kappa_{10}[f(b, c) - f(a, c)] + \kappa_{11}h[p(b, c) + p(a, c)] + \kappa_{12}h^2[r(b, c) - r(a, c)] \\ h^2r(x, c) &= \kappa_{20}[f(b, c) + f(a, c)] + \kappa_{21}h[p(b, c) - p(a, c)] + \kappa_{22}h^2[r(b, c) + r(a, c)] \end{aligned} \right\}. \quad (4)$$

Then we use the scheme HC^1 to build $q = f_y, s = q_x = f_{xy}$:

$$\left. \begin{aligned} q(x, c) &= \lambda_{00}[q(b, c) + q(a, c)] + \lambda_{01}h[s(b, c) - s(a, c)] \\ hs(x, c) &= \lambda_{10}[q(b, c) - q(a, c)] + \lambda_{11}h[s(b, c) + s(a, c)] \end{aligned} \right\} \quad (5)$$

and finally for $t = f_{y^2}$:

$$t(x, c) = [t(b, c) + t(a, c)]/2. \quad (6)$$

Similar formulae give the values of the functions on $(x, y = d)$.

Case 2: $(x, y) \in \mathcal{P}_n \times (\mathcal{P}_{n+1} \setminus \mathcal{P}_n)$.

We set $a = x, c = y - h/2, d = y + h/2$.

$$\left. \begin{aligned} f(a, y) &= \kappa_{00}[f(a, d) + f(a, c)] + \kappa_{01}h[q(a, d) - q(a, c)] + \kappa_{02}h^2[t(a, d) + t(a, c)] \\ hq(a, y) &= \kappa_{10}[f(a, d) - f(a, c)] + \kappa_{11}h[q(a, d) + q(a, c)] + \kappa_{12}h^2[t(a, d) - t(a, c)] \\ h^2t(a, y) &= \kappa_{20}[f(a, d) + f(a, c)] + \kappa_{21}h[q(a, d) - q(a, c)] + \kappa_{22}h^2[t(a, d) + t(a, c)] \end{aligned} \right\},$$

$$\left. \begin{aligned} p(a, y) &= \lambda_{00}[p(a, d) + p(a, c)] + \lambda_{01}h[s(a, d) - s(a, c)] \\ hs(a, y) &= \lambda_{10}[p(a, d) - p(a, c)] + \lambda_{11}h[s(a, d) + s(a, c)] \end{aligned} \right\},$$

$$r(a, y) = [r(a, d) + r(a, c)]/2.$$

Similar formulae give the values of the functions on $(x = b, y)$.

Case 3: $(x, y) \in (\mathcal{P}_{n+1} \setminus \mathcal{P}_n) \times (\mathcal{P}_{n+1} \setminus \mathcal{P}_n)$.

We set $a = x - h/2, b = x + h/2, c = y - h/2, d = y + h/2$ and we denote by Q the square $[a, b] \times [c, d]$. We consider two operators $\partial/\partial u := \partial/\partial x + \partial/\partial y$ and $\partial/\partial v := -\partial/\partial x + \partial/\partial y$. These operators and their compositions will follow the usual notation $f_u = (\partial/\partial u)f, f_v = (\partial/\partial v)f, f_{u^2} = (\partial/\partial u)^2 f, f_{uv} = (\partial/\partial u)(\partial/\partial v)f, f_{v^2} = (\partial/\partial v)^2 f$.

Firstly we define $f(x, y)$ by using the scheme HC^2 defined in (3) along both diagonals of Q . We define $f_1(x, y) = \phi(1/2)$ by applying the scheme HC^2 to the function $\phi(t) = f(a + th, c + th)$ and similarly $f_2(x, y) = \psi(1/2)$ by applying the scheme HC^2 for the function $\psi(t) = f(b - th, c + th)$.

$$\begin{aligned} f_1(x, y) &= \kappa_{00}[f(b, d) + f(a, c)] + \kappa_{01}h[f_u(b, d) - f_u(a, c)] + \kappa_{02}h^2[f_{u^2}(b, d) + f_{u^2}(a, c)], \\ f_2(x, y) &= \kappa_{00}[f(a, d) + f(b, c)] + \kappa_{01}h[f_v(a, d) - f_v(b, c)] + \kappa_{02}h^2[f_{v^2}(a, d) + f_{v^2}(b, c)]. \end{aligned}$$

We now define $f(x, y) = [f_1(x, y) + f_2(x, y)]/2$ and after using the symbolic rules: $f_{u^2} = f_{x^2} + 2f_{xy} + f_{y^2}, f_{v^2} = f_{x^2} - 2f_{xy} + f_{y^2}$, we get

$$\begin{aligned} 2f(x, y) &= \kappa_{00}[f(b, d) + f(a, c) + f(a, d) + f(b, c)] \\ &\quad + \kappa_{01}h[p(b, d) - p(a, c) - p(a, d) + p(b, c)] \\ &\quad + \kappa_{01}h[q(b, d) - q(a, c) + q(a, d) - q(b, c)] \\ &\quad + \kappa_{02}h^2[r(b, d) + r(a, c) + r(a, d) + r(b, c)] \\ &\quad + 2\kappa_{02}h^2[s(b, d) + s(a, c) - s(a, d) - s(b, c)] \\ &\quad + \kappa_{02}h^2[t(b, d) + t(a, c) + t(a, d) + t(b, c)]. \end{aligned}$$

Secondly we define $p(x, y)$ and $q(x, y)$ by using the scheme HC^2 in (3) along both diagonals of Q in order to get the values $f_u(x, y) = \phi'(1/2)$ and $f_v(x, y) = \psi'(1/2)$.

$$\begin{aligned} hf_u(x, y) &= \kappa_{10}[f(b, d) - f(a, c)] + \kappa_{11}h[f_u(b, d) + f_u(a, c)] + \kappa_{12}h^2[f_{u^2}(b, d) - f_{u^2}(a, c)], \\ hf_v(x, y) &= \kappa_{10}[f(a, d) - f(b, c)] + \kappa_{11}h[f_v(a, d) + f_v(b, c)] + \kappa_{12}h^2[f_{v^2}(a, d) - f_{v^2}(b, c)]. \end{aligned}$$

After using the symbolic rules: $f_{u^2} = f_{x^2} + 2f_{xy} + f_{y^2}, f_{v^2} = f_{x^2} - 2f_{xy} + f_{y^2}$,

$f_x = (f_u - f_v)/2$, $f_y = (f_u + f_v)/2$, we get

$$\begin{aligned}
2hp(x, y) &= \kappa_{10}[f(b, d) - f(a, c) - f(a, d) + f(b, c)] \\
&\quad + \kappa_{11}h[p(b, d) + p(a, c) + p(a, d) + p(b, c)] \\
&\quad + \kappa_{11}h[q(b, d) + q(a, c) - q(a, d) - q(b, c)] \\
&\quad + \kappa_{12}h^2[r(b, d) - r(a, c) - r(a, d) + r(b, c)] \\
&\quad + 2\kappa_{12}h^2[s(b, d) - s(a, c) + s(a, d) - s(b, c)] \\
&\quad + \kappa_{12}h^2[t(b, d) - t(a, c) - t(a, d) + t(b, c)]
\end{aligned}$$

and

$$\begin{aligned}
2hq(x, y) &= \kappa_{10}[f(b, d) - f(a, c) + f(a, d) - f(b, c)] \\
&\quad + \kappa_{11}h[p(b, d) + p(a, c) - p(a, d) - p(b, c)] \\
&\quad + \kappa_{11}h[q(b, d) + q(a, c) + q(a, d) + q(b, c)] \\
&\quad + \kappa_{12}h^2[r(b, d) - r(a, c) + r(a, d) - r(b, c)] \\
&\quad + 2\kappa_{12}h^2[s(b, d) - s(a, c) - s(a, d) + s(b, c)] \\
&\quad + \kappa_{12}h^2[t(b, d) - t(a, c) + t(a, d) - t(b, c)].
\end{aligned}$$

Thirdly we define $r(x, y), t(x, y)$ by using the scheme HC^2 in (3) along both diagonals of Q in order to get the values $f_{u^2}(x, y) = \phi''(1/2)$ and $f_{v^2}(x, y) = \psi''(1/2)$. For defining $s(x, y)$, we apply the scheme HC^1 (2) to the functions $\theta(t) = f_v(a + th, c + th)$ and $\tau(t) = f_u(b - th, c + th)$. We set $f_{uv}(x, y) = [\theta'(1/2) + \tau'(1/2)]/2$.

$$\begin{aligned}
h^2 f_{u^2}(x, y) &= \kappa_{20}[f(b, d) + f(a, c)] + \kappa_{21}h[f_u(b, d) - f_u(a, c)] + \kappa_{22}h^2[f_{u^2}(b, d) + f_{u^2}(a, c)], \\
h^2 f_{v^2}(x, y) &= \kappa_{20}[f(a, d) + f(b, c)] + \kappa_{21}h[f_v(a, d) - f_v(b, c)] + \kappa_{22}h^2[f_{v^2}(a, d) + f_{v^2}(b, c)], \\
2h^2 f_{uv}(x, y) &= \lambda_{10}h[f_v(b, d) - f_v(a, c)] + \lambda_{11}h^2[f_{uv}(b, d) + f_{uv}(a, c)] \\
&\quad + \lambda_{10}h[f_u(a, d) - f_u(b, c)] + \lambda_{11}h^2[f_{uv}(a, d) + f_{uv}(b, c)].
\end{aligned}$$

After using the symbolic rules: $f_{u^2} = f_x^2 + 2f_{xy} + f_y^2$, $f_{v^2} = f_x^2 - 2f_{xy} + f_y^2$, $f_x^2 = (f_{u^2} - 2f_{uv} + f_{v^2})/4$, $f_{xy} = (f_{u^2} - f_{v^2})/4$, $f_y^2 = (f_{u^2} + 2f_{uv} + f_{v^2})/4$, we get

$$\begin{aligned}
4h^2 r(x, y) &= \kappa_{20}[f(b, d) + f(a, c) + f(a, d) + f(b, c)] \\
&\quad + (\kappa_{21} + \lambda_{10})h[p(b, d) - p(a, c) - p(a, d) + p(b, c)] \\
&\quad + (\kappa_{21} - \lambda_{10})h[q(b, d) - q(a, c) + q(a, d) - q(b, c)] \\
&\quad + (\kappa_{22} + \lambda_{11})h^2[r(b, d) + r(a, c) + r(a, d) + r(b, c)] \\
&\quad + 2\kappa_{22}h^2[s(b, d) + s(a, c) - s(a, d) - s(b, c)] \\
&\quad + (\kappa_{22} - \lambda_{11})h^2[t(b, d) + t(a, c) + t(a, d) + t(b, c)],
\end{aligned}$$

$$\begin{aligned}
4h^2s(x, y) = & \kappa_{20}[f(b, d) + f(a, c) - f(a, d) - f(b, c)] \\
& + \kappa_{21}h[p(b, d) - p(a, c) + p(a, d) - p(b, c)] \\
& + \kappa_{21}h[q(b, d) - q(a, c) - q(a, d) + q(b, c)] \\
& + \kappa_{22}h^2[r(b, d) + r(a, c) - r(a, d) - r(b, c)] \\
& + 2\kappa_{22}h^2[s(b, d) + s(a, c) + s(a, d) + s(b, c)] \\
& + \kappa_{22}h^2[t(b, d) + t(a, c) - t(a, d) - t(b, c)],
\end{aligned}$$

and

$$\begin{aligned}
4h^2t(x, y) = & \kappa_{20}[f(b, d) + f(a, c) + f(a, d) + f(b, c)] \\
& + (\kappa_{21} - \lambda_{10})h[p(b, d) - p(a, c) - p(a, d) + p(b, c)] \\
& + (\kappa_{21} + \lambda_{10})h[q(b, d) - q(a, c) + q(a, d) - q(b, c)] \\
& + (\kappa_{22} - \lambda_{11})h^2[r(b, d) + r(a, c) + r(a, d) + r(b, c)] \\
& + 2\kappa_{22}h^2[s(b, d) + s(a, c) - s(a, d) - s(b, c)] \\
& + (\kappa_{22} + \lambda_{11})h^2[t(b, d) + t(a, c) + t(a, d) + t(b, c)].
\end{aligned}$$

We now exhibit the mask A of the Hermite subdivision scheme HD^2 of matrix parameters (Λ, K) . $A(0, 0) = D$.

$$A(-\epsilon, 0) = DS_1(\epsilon) \begin{pmatrix} \kappa_{00} & \kappa_{01} & 0 & \kappa_{02} & 0 & 0 \\ \kappa_{10} & \kappa_{11} & 0 & \kappa_{12} & 0 & 0 \\ 0 & 0 & \lambda_{00} & 0 & \lambda_{01} & 0 \\ \kappa_{20} & \kappa_{21} & 0 & \kappa_{22} & 0 & 0 \\ 0 & 0 & \lambda_{10} & 0 & \lambda_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} S_1(\epsilon), \quad (7)$$

where $S_1(\epsilon) = \text{diag}(1, \epsilon, 1, 1, \epsilon, 1)$ for $\epsilon = \pm 1$,

$$A(0, -\epsilon) = DS_2(\epsilon) \begin{pmatrix} \kappa_{00} & 0 & \kappa_{01} & 0 & 0 & \kappa_{02} \\ 0 & \lambda_{00} & 0 & 0 & \lambda_{01} & 0 \\ \kappa_{10} & 0 & \kappa_{11} & 0 & 0 & \kappa_{12} \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & \lambda_{10} & 0 & 0 & \lambda_{11} & 0 \\ \kappa_{20} & 0 & \kappa_{21} & 0 & 0 & \kappa_{22} \end{pmatrix} S_2(\epsilon), \quad (8)$$

where $S_2(\epsilon) = \text{diag}(1, 1, \epsilon, 1, \epsilon, 1)$ for $\epsilon = \pm 1$,

$$A(-\epsilon_1, -\epsilon_2) = DR_1S_3(\epsilon_1, \epsilon_2)B_1S_3(\epsilon_1, \epsilon_2), \quad (9)$$

where $R_1 = \text{diag}(1/2, 1/2, 1/2, 1/4, 1/4, 1/4)$ $S_3(\epsilon_1, \epsilon_2) = \text{diag}(1, \epsilon_1, \epsilon_2, 1, \epsilon_1\epsilon_2, 1)$, for $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ and

$$B_1 = \begin{pmatrix} \kappa_{00} & \kappa_{01} & \kappa_{01} & \kappa_{02} & 2\kappa_{02} & \kappa_{02} \\ \kappa_{10} & \kappa_{11} & \kappa_{11} & \kappa_{12} & 2\kappa_{12} & \kappa_{12} \\ \kappa_{10} & \kappa_{11} & \kappa_{11} & \kappa_{12} & 2\kappa_{12} & \kappa_{12} \\ \kappa_{20} & \kappa_{21} + \lambda_{10} & \kappa_{21} - \lambda_{10} & \kappa_{22} + \lambda_{11} & 2\kappa_{22} & \kappa_{22} - \lambda_{11} \\ \kappa_{20} & \kappa_{21} & \kappa_{21} & \kappa_{22} & 2\kappa_{22} & \kappa_{22} \\ \kappa_{20} & \kappa_{21} - \lambda_{10} & \kappa_{21} + \lambda_{10} & \kappa_{22} - \lambda_{11} & 2\kappa_{22} & \kappa_{22} + \lambda_{11} \end{pmatrix}.$$

Finally for any other $(i, j) \in \mathbb{Z}^2$, $A(i, j) = 0$.

3.2 The recursive scheme HR^2

An algorithm HR^1 which gives a C^1 Hermite interpolant on a rectangle has been proposed by Dubuc and Merrien [1]. We now describe an algorithm HR^2 to obtain C^2 interpolation. At the beginning of the construction, the only data that are known about these six functions f, p, q, r, s, t are their values at the vertices of $R = [0, 1]^2$. As in the previous subsection, the construction depends on the 2 square matrix parameters Λ and K of respective orders 2 and 3. For $n = 0, 1, 2, \dots$, let us denote again by \mathcal{P}_n the regular partition of I into 2^n subintervals. We proceed by induction on n and we assume that f, p, q, r, s, t are already known on the mesh $\mathcal{P}_n \times \mathcal{P}_n$. We will then define these functions on $(\mathcal{P}_{n+1} \times \mathcal{P}_{n+1}) \setminus (\mathcal{P}_n \times \mathcal{P}_n)$. Let $h = |I|/2^n$. (see Figure 2).

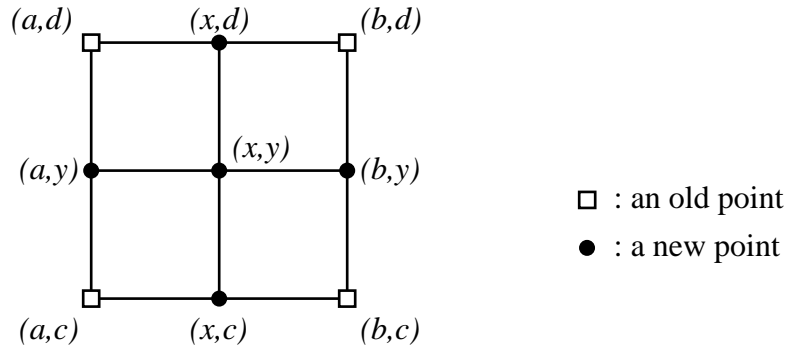


Fig. 2. Dyadic points at steps n and $n + 1$.

For the two first cases $(x, y) \in (\mathcal{P}_{n+1} \setminus \mathcal{P}_n) \times \mathcal{P}_n$ and $(x, y) \in \mathcal{P}_n \times (\mathcal{P}_{n+1} \setminus \mathcal{P}_n)$, the computation of f, p, q, r, s, t on (x, y) is done by the same formulae as in Cases 1 and 2 of the previous subsection. In order to compute f, p, q, r, s, t on (x, y) when $(x, y) \in (\mathcal{P}_{n+1} \setminus \mathcal{P}_n) \times (\mathcal{P}_{n+1} \setminus \mathcal{P}_n)$, we proceed as follows. We set $a = x - h/2, b = x + h/2, c = y - h/2, d = y + h/2$ and we denote by Q the square $[a, b] \times [c, d]$.

Firstly we apply the scheme HC^2 (3) to the function $\phi(\cdot) = f(a + h\cdot, y)$ for defining a first approximation $f_1(x, y) = \phi(1/2)$ of $f(x, y)$ together with the approximations $p(x, y) = \phi'(1/2), r(x, y) = \phi''(1/2)$. Notice first that f, p, r have been defined on (a, y) and (b, y) in Case 2.

$$\left. \begin{aligned} f_1(x, y) &= \kappa_{00}[f(b, y) + f(a, y)] + \kappa_{01}h[p(b, y) - p(a, y)] + \kappa_{02}h^2[r(b, y) + r(a, y)] \\ hp(x, y) &= \kappa_{10}[f(b, y) - f(a, y)] + \kappa_{11}h[p(b, y) + p(a, y)] + \kappa_{12}h^2[r(b, y) - r(a, y)] \\ h^2r(x, y) &= \kappa_{20}[f(b, y) + f(a, y)] + \kappa_{21}h[p(b, y) - p(a, y)] + \kappa_{22}h^2[r(b, y) + r(a, y)] \end{aligned} \right\}.$$

Secondly by symmetry, we apply the scheme HC^2 (3) to the function $\psi(\cdot) = f(x, c + h\cdot)$ for defining a second approximation $f_2(x, y) = \psi(1/2)$ of $f(x, y)$ together with $q(x, y) = \psi'(1/2), t(x, y) = \psi''(1/2)$. Notice again that f, q, t have been defined on (x, c) and (x, d) in Case 1.

$$\left. \begin{aligned} f_2(x, y) &= \kappa_{00}[f(x, d) + f(x, c)] + \kappa_{01}h[q(x, d) - q(x, c)] + \kappa_{02}h^2[t(x, d) + t(x, c)] \\ hq(x, y) &= \kappa_{10}[f(x, d) - f(x, c)] + \kappa_{11}h[q(x, d) + q(x, c)] + \kappa_{12}h^2[t(x, d) - t(x, c)] \\ h^2t(x, y) &= \kappa_{20}[f(x, d) + f(x, c)] + \kappa_{21}h[q(x, d) - q(x, c)] + \kappa_{22}h^2[t(x, d) + t(x, c)] \end{aligned} \right\}.$$

We now define $f(x, y) = [f_1(x, y) + f_2(x, y)]/2$ and we get

$$\begin{aligned} 2f(x, y) &= 2\kappa_{00}^2[f(b, d) + f(b, c) + f(a, d) + f(a, c)] \\ &\quad + \kappa_{01}(\lambda_{00} + \kappa_{00})h[p(b, d) + p(b, c) - p(a, d) - p(a, c)] \\ &\quad + \kappa_{01}(\lambda_{00} + \kappa_{00})h[q(b, d) - q(b, c) + q(a, d) - q(a, c)] \\ &\quad + \kappa_{02}(\kappa_{00} + 1/2)h^2[r(b, d) + r(b, c) + r(a, d) + r(a, c)] \\ &\quad + 2\kappa_{01}\lambda_{01}h^2[s(b, d) - s(b, c) - s(a, d) + s(a, c)] \\ &\quad + \kappa_{02}(\kappa_{00} + 1/2)h^2[t(b, d) + t(a, d) + t(b, c) + t(a, c)]. \end{aligned}$$

$$\begin{aligned}
hp(x, y) = & \kappa_{10}\kappa_{00}[f(b, d) + f(b, c) - f(a, d) - f(a, c)] \\
& + \kappa_{11}\lambda_{00}h[p(b, d) + p(b, c) + p(a, d) + p(a, c)] \\
& + \kappa_{10}\kappa_{01}h[q(b, d) - q(b, c) - q(a, d) + q(a, c)] \\
& + \kappa_{12}/2h^2[r(b, d) + r(b, c) - r(a, d) - r(a, c)] \\
& + \kappa_{11}\lambda_{01}h^2[s(b, d) - s(b, c) + s(a, d) - s(a, c)] \\
& + \kappa_{10}\kappa_{02}h^2[t(b, d) + t(b, c) - t(a, d) - t(a, c)].
\end{aligned}$$

$$\begin{aligned}
hq(x, y) = & \kappa_{10}\kappa_{00}[f(b, d) - f(b, c) + f(a, d) - f(a, c)] \\
& + \kappa_{10}\kappa_{01}h[p(b, d) - p(b, c) - p(a, d) + p(a, c)] \\
& + \kappa_{11}\lambda_{00}h[q(b, d) + q(b, c) + q(a, d) + q(a, c)] \\
& + \kappa_{10}\kappa_{02}h^2[r(b, d) - r(b, c) + r(a, d) - r(a, c)] \\
& + \kappa_{11}\lambda_{01}h^2[s(b, d) + s(b, c) - s(a, d) - s(a, c)] \\
& + \kappa_{12}/2h^2[t(b, d) - t(b, c) + t(a, d) - t(a, c)].
\end{aligned}$$

$$\begin{aligned}
h^2r(x, y) = & \kappa_{20}\kappa_{00}[f(b, d) + f(b, c) + f(a, d) + f(a, c)] \\
& + \kappa_{21}\lambda_{00}h[p(b, d) + p(b, c) - p(a, d) - p(a, c)] \\
& + \kappa_{20}\kappa_{01}h[q(b, d) - q(b, c) + q(a, d) - q(a, c)] \\
& + \kappa_{22}/2h^2[r(b, d) + r(b, c) + r(a, d) + r(a, c)] \\
& + \kappa_{21}\lambda_{01}h^2[s(b, d) - s(b, c) - s(a, d) + s(a, c)] \\
& + \kappa_{20}\kappa_{02}h^2[t(b, d) + t(b, c) + t(a, d) + t(a, c)].
\end{aligned}$$

$$\begin{aligned}
h^2t(x, y) = & \kappa_{20}\kappa_{00}[f(b, d) + f(b, c) + f(a, d) + f(a, c)] \\
& + \kappa_{20}\kappa_{01}h[p(b, d) + p(b, c) - p(a, d) - p(a, c)] \\
& + \kappa_{21}\lambda_{00}h[q(b, d) - q(b, c) + q(a, d) - q(a, c)] \\
& + \kappa_{20}\kappa_{02}h^2[r(b, d) + r(b, c) + r(a, d) + r(a, c)] \\
& + \kappa_{21}\lambda_{01}h^2[s(b, d) - s(b, c) - s(a, d) + s(a, c)] \\
& + \kappa_{22}/2h^2[t(b, d) + t(b, c) + t(a, d) + t(a, c)].
\end{aligned}$$

Thirdly we apply the scheme HC^1 in (2) to the functions $\theta(\cdot) = f_y(a + h\cdot, y)$ and $\tau(\cdot) = f_x(x, c + h\cdot)$. We set $s_1(x, y) = \theta'(1/2)$, $s_2(x, y) = \tau'(1/2)/2$.

$$\begin{aligned}
h^2 s_1(x, y) &= \lambda_{10} h[q(b, y) - q(a, y)] + \lambda_{11} h^2[s(b, y) + s(a, y)], \\
h^2 s_2(x, y) &= \lambda_{10} h[p(x, d) - p(x, c)] + \lambda_{11} h^2[s(x, d) + s(x, c)].
\end{aligned}$$

We now define $s(x, y) = [s_1(x, y) + s_2(x, y)]/2$ and we get

$$\begin{aligned}
2h^2 s(x, y) &= 2\lambda_{10}\kappa_{10}[f(b, d) - f(b, c) - f(a, d) + f(a, c)] \\
&\quad + \lambda_{10}(\kappa_{11} + \lambda_{11})h[p(b, d) - p(b, c) + p(a, d) - p(a, c)] \\
&\quad + \lambda_{10}(\kappa_{11} + \lambda_{11})h[q(b, d) + q(b, c) - q(a, d) - q(a, c)] \\
&\quad + \lambda_{10}\kappa_{12}h^2[r(b, d) - r(b, c) - r(a, d) + r(a, c)] \\
&\quad + 2\lambda_{11}^2 h^2[s(b, d) + s(b, c) + s(a, d) + s(a, c)] \\
&\quad + \lambda_{10}\kappa_{12}h^2[t(b, d) - t(b, c) - t(a, d) + t(a, c)].
\end{aligned}$$

For the mask of the Hermite subdivision scheme HR^2 , $A(0, 0) = D$, the matrices $A(\pm 1, 0)$ are given by (7) and the matrices $A(0, \pm 1)$, by (8). Then

$$A(-\epsilon_1, -\epsilon_2) = DR_2 S_3(\epsilon_1, \epsilon_2) B_2 S_3(\epsilon_1, \epsilon_2), \quad (10)$$

where $R_2 = \text{diag}(1/2, 1, 1, 1, 1/2, 1)$, $S_3(\epsilon_1, \epsilon_2) = \text{diag}(1, \epsilon_1, \epsilon_2, 1, \epsilon_1\epsilon_2, 1)$, for $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ and B_2 is the matrix

$$\begin{pmatrix}
2\kappa_{00}^2 & \kappa_{01}(\lambda_{00} + \kappa_{00}) & \kappa_{01}(\lambda_{00} + \kappa_{00}) & \kappa_{02}(\kappa_{00} + 1/2) & 2\kappa_{01}\lambda_{01} & \kappa_{02}(\kappa_{00} + 1/2) \\
\kappa_{10}\kappa_{00} & \kappa_{11}\lambda_{00} & \kappa_{10}\kappa_{01} & \kappa_{12}/2 & \kappa_{11}\lambda_{01} & \kappa_{10}\kappa_{02} \\
\kappa_{10}\kappa_{00} & \kappa_{10}\kappa_{01} & \kappa_{11}\lambda_{00} & \kappa_{10}\kappa_{02} & \kappa_{11}\lambda_{01} & \kappa_{12}/2 \\
\kappa_{20}\kappa_{00} & \kappa_{21}\lambda_{00} & \kappa_{20}\kappa_{01} & \kappa_{22}/2 & \kappa_{21}\lambda_{01} & \kappa_{20}\kappa_{02} \\
2\lambda_{10}\kappa_{10} & \lambda_{10}(\lambda_{11} + \kappa_{11}) & \lambda_{10}(\lambda_{11} + \kappa_{11}) & \lambda_{10}\kappa_{12} & 2\lambda_{11}^2 & \lambda_{10}\kappa_{12} \\
\kappa_{20}\kappa_{00} & \kappa_{20}\kappa_{01} & \kappa_{21}\lambda_{00} & \kappa_{20}\kappa_{02} & \kappa_{21}\lambda_{01} & \kappa_{22}/2
\end{pmatrix}.$$

For any other $(i, j) \in \mathbb{Z}^2$, $A(i, j) = 0$.

4 Convergence of Hermite subdivision schemes

In this section, we recall the definition of the convergence of Hermite interpolatory subdivision schemes. Convergent Hermite interpolatory subdivision schemes always provide a solution to a refinement equation.

Definition 1 Let f_0 be a function from \mathbb{Z}^s to \mathbb{R}^M , the initial data of an interpolatory Hermite subdivision scheme \mathcal{H} of type (r, s) . If $m \geq r$ and f_n is the corresponding sequence of refinements of \mathcal{H} , then the scheme is called C^m for f_0 if there exists a C^m -function $\phi : \mathbb{R}^s \rightarrow \mathbb{R}$ such that

$$f_n^\mu(\alpha) = \partial^\mu \phi(\alpha/2^n)$$

for any s -tuple $\mu \in M$ and for any $\alpha \in \mathbb{Z}^s$. The scheme is C^m if it is C^m for any initial data $f_0 : \mathbb{Z}^s \rightarrow \mathbb{R}^M$.

Definition 2 The basic matrix refinements of a Hermite subdivision scheme of type (r, s) are the recursive sequence of matrix functions Φ_n ($\Phi_n : \mathbb{Z}^s \rightarrow \mathbb{R}^{M \times M}$):

$$D^{n+1}\Phi_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta)D^n\Phi_n(\beta), \alpha \in \mathbb{Z}^s, n \in \mathbb{N}_0, \quad (11)$$

where $\Phi_0(0)$ is the identity matrix of order $\#M$ and $\Phi_0(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{0\}$.

Remark 1 There is a close link between the first matrix refinement Φ_1 and the mask of a Hermite scheme: $A(\alpha) = D\Phi_1(\alpha)$.

From Formula (1), we deduce that for any initial data $f_0(i)$ we have

$$f_n(\alpha) = \sum_{j \in \mathbb{Z}^s} \Phi_n(\alpha - 2^n \beta) f_0(\beta), \alpha \in \mathbb{Z}^s, n \in \mathbb{N}_0.$$

Definition 3 If Φ_n is the sequence of basic matrix refinements of a C^r Hermite subdivision scheme, then there exist $\#M$ functions $\phi_\mu : \mathbb{R}^s \rightarrow \mathbb{R}$ where $\mu \in M$ such that the (μ, ν) -entry of $\Phi_n(\alpha)$ is $\partial^\mu \phi_\nu(\alpha/2^n)$. We define the basic matrix function of a C^r Hermite subdivision scheme as the matrix $\Phi(x)$ whose (μ, ν) -th component is $\partial^\mu \phi_\nu(x)$.

Lemma 1 If Φ_n is the sequence of basic matrix refinements of a Hermite subdivision scheme of type (r, s) , then for $n = 0, 1, \dots$,

$$D\Phi_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} \Phi_n(\alpha - 2^n \beta) A(\beta). \quad (12)$$

Proof: We will proceed by induction on n . The case $n = 0$ is Remark 1. If $n \geq 0$, we first use Equation (11) at step n in Definition 2 then the hypotheses of the induction. We change the order of the finite summations and we set $\beta = \beta' + 2^{n-1}\gamma$. Last, we use again Equation (11) at step $n - 1$ and we get

$$\begin{aligned}
D^{n+1}\Phi_{n+1}(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta) D^n \Phi_n(\beta) \\
&= \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta) D^{n-1} \sum_{\gamma \in \mathbb{Z}^s} \Phi_{n-1}(\beta - 2^{n-1}\gamma) A(\gamma) \\
&= \sum_{\gamma \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta) D^{n-1} \Phi_{n-1}(\beta - 2^{n-1}\gamma) A(\gamma) \\
&= \sum_{\gamma \in \mathbb{Z}^s} \sum_{\beta' \in \mathbb{Z}^s} A(\alpha - 2\beta' - 2^n\gamma) D^{n-1} \Phi_{n-1}(\beta') A(\gamma) \\
&= \sum_{\gamma \in \mathbb{Z}^s} D^n \Phi_n(\alpha - 2^n\gamma) A(\gamma).
\end{aligned}$$

Therefore, we obtain Equation (12). \square

Theorem 2 *If Φ is the basic matrix function of a C^r Hermite interpolatory subdivision scheme of type (r, s) , then*

$$(\forall x \in \mathbb{R}^s) D\Phi(x/2) = \sum_{\beta \in \mathbb{Z}^s} \Phi(x - \beta) A(\beta). \quad (13)$$

Proof: Since the Hermite subdivision scheme is interpolatory and C^r , we obtain $\Phi_n(\alpha) = \Phi(\alpha/2^n)$ and Equation (13) is satisfied for every dyadic point $x = \alpha/2^n$. By continuity Equation (13) is true for every point $x \in \mathbb{R}^s$. \square

Corollary 3 *If $\phi(x)$ is the first row of the basic matrix function $\Phi(x)$ of a C^r Hermite subdivision scheme of type (r, s) with A as mask, then the refinement equation*

$$\phi(x/2) = \sum_{\beta \in \mathbb{Z}^s} \phi(x - \beta) A(\beta), \quad \forall x \in \mathbb{R}^s. \quad (14)$$

is satisfied.

We say that a vector of functions $\phi = (\phi_\nu)_{\nu \in M}$ is a *basic Hermite interpolant of type (r, s)* if all functions $\phi_\nu : \mathbb{R}^s \rightarrow \mathbb{R}$ belongs to C^r and satisfy

$$\partial^\mu \phi_\nu(\alpha) = \delta(\nu - \mu) \delta(\alpha), \quad \mu, \nu \in M, \alpha \in \mathbb{Z}^s. \quad (15)$$

where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus \{0\}$.

Theorem 4 *Let A be the mask of a Hermite subdivision scheme \mathcal{H} of type (r, s) and let $\phi = (\phi_\nu)_{\nu \in M}$ be a basic Hermite interpolant of the same type. If ϕ is a solution of the Equation (14), then \mathcal{H} is interpolatory. Moreover, \mathcal{H} is C^r and the (μ, ν) -entry of its basic matrix function Φ is $\partial^\mu \phi_\nu$.*

Proof: Let $\phi = (\phi_\nu)_{\nu \in M}$ be a basic Hermite interpolant of type (r, s) , solution of (14). We take the μ -partial derivative of both sides of this equation:

$$\partial^\mu \phi(x/2)/2^{|\mu|} = \sum_{\beta \in \mathbb{Z}^s} \partial^\mu \phi(x - \beta)A(\beta), \quad \forall x \in \mathbb{R}^s, \mu \in M. \quad (16)$$

If in the last equation, we set $x = 2\alpha$, we obtain

$$A(2\alpha) = \delta(\alpha)D, \quad \alpha \in \mathbb{Z}^s.$$

where $D = \text{diag}(1/2^\mu)_{\mu \in M}$. Then \mathcal{H} is interpolatory.

Let $\Phi_n : \mathbb{R}^s \rightarrow \mathbb{R}$ be the sequence of matrix refinements of \mathcal{H} . We will show by induction on n that

$$\Phi_n(\alpha) = (\partial^\mu \phi_\nu(\alpha/2^n))_{\mu, \nu \in M}, \quad \alpha \in \mathbb{Z}^s. \quad (17)$$

The case $n = 0$ is trivial.

If $n \geq 0$, then we first use (11) at step n in Definition 2 and the hypotheses of the induction (17), secondly we set $x = \alpha/2^n$ in (16) and we get

$$\begin{aligned} D\Phi_{n+1}(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} (\partial^\mu \phi(\alpha/2^n - \beta))_{\mu, \nu \in M} A(\beta) \\ &= (\partial^\mu \phi(\alpha/2^{n+1})/2^{|\mu|})_{\mu, \nu \in M}. \end{aligned}$$

Equation (17) is satisfied for the next integer $n + 1$.

The matrix function $\Phi(x) = (\partial^\mu \phi_\nu(x))_{\mu, \nu \in M}$ is continuous and $\Phi_n(\alpha) = \Phi(\alpha/2^n)$, $\alpha \in \mathbb{Z}^s$. The Hermite subdivision scheme \mathcal{H} is C^r . \square

5 Smoothness of Hermite interpolatory subdivision schemes

In this section, we consider the *refinement equation*

$$\phi(x/2) = \sum_{\beta \in \mathbb{Z}^s} \phi(x - \beta)A(\beta), \quad \forall x \in \mathbb{R}^s \quad (18)$$

where $\phi = (\phi_1, \dots, \phi_m)$ is a $1 \times m$ row vector of generalized functions, called a *refinable vector function*, and $A : \mathbb{Z}^s \rightarrow \mathbb{R}^{m \times m}$ is a finitely supported matrix function on \mathbb{Z}^s , called the (*matrix refinement*) *mask*.

We denote Π_k the set of all polynomials of total degree at most k . Let $(\ell_0(\mathbb{Z}^s))^{m \times n}$ denote the set of all finitely supported sequences of $m \times n$ matrices. A mask A with multiplicity m is a finitely supported sequence of $m \times m$

matrices on \mathbb{Z}^s ; that is, $A \in (\ell_0(\mathbb{Z}^s))^{m \times m}$. Recall that $M = M(r, s) := \{(\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s : \mu_1 + \mu_2 + \dots + \mu_s \leq r\}$. For two nonnegative integers k and r , we define a subset $\mathcal{P}_{k,r}$ of $(\Pi_k)^{(\#M) \times 1}$ by

$$\mathcal{P}_{k,r} := \{\partial^{\leq r} p : p \in \Pi_k\} \quad \text{with} \quad \partial^{\leq r} p := (\partial^\mu p)_{\mu \in M}, \quad (19)$$

and we define the *subdivision operator* S_A by

$$[S_A p](\alpha) = \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta) p(\beta), \quad \alpha \in \mathbb{Z}^s, p \in \mathcal{P}_{k,r}, \quad (20)$$

where we identify $p \in \mathcal{P}_{k,r}$ with the sequence $(p(\beta))_{\beta \in \mathbb{Z}^s}$ which is induced by p .

Now the recursive refinements in Equation 1 can be rewritten as

$$D^{n+1} f_{n+1} = S_A [D^n f_n], \quad n \in \mathbb{N}_0. \quad (21)$$

Let A denote the mask in a Hermite interpolatory subdivision scheme \mathcal{H} of type (r, s) . For any $p \in \Pi_r$, we define the initial data f_0 by $f_0(\alpha) = [\partial^{\leq r} p](\alpha)$, $\alpha \in \mathbb{Z}^s$; that is, $f_0^\mu(\alpha) = [\partial^\mu p](\alpha)$, $\mu \in M$, $\alpha \in \mathbb{Z}^s$. It is very natural to require that the Hermite interpolatory subdivision scheme should exactly reproduce all the polynomials in Π_r , that is,

$$f_{n+1}(\alpha) = [\partial^{\leq r} p](2^{-n-1}\alpha) \quad \forall \alpha \in \mathbb{Z}^s, n \in \mathbb{N}_0, p \in \Pi_r.$$

By (21), the above relation is equivalent to the requirement

$$S_A [D^n (\partial^{\leq r} p)(2^{-n} \cdot)] = D^{n+1} (\partial^{\leq r} p)(2^{-n-1} \cdot), \quad n \in \mathbb{N}_0, p \in \Pi_r.$$

Now it is easy to see that it suffices to require that

$$S_A [\partial^{\leq r} p] = D(\partial^{\leq r} p)(2^{-1} \cdot) = \partial^{\leq r} [p(2^{-1} \cdot)] \quad \forall p \in \Pi_r. \quad (22)$$

The above equation motivates us to define sum rules for Hermite interpolatory subdivision schemes. We say that a mask A with multiplicity $\#M(r, s)$ satisfies the *interpolatory sum rules* of order $k + 1$ if

$$S_A [\partial^{\leq r} p] = \partial^{\leq r} [p(2^{-1} \cdot)] \quad \forall p \in \Pi_k. \quad (23)$$

A more general definition of sum rules has been given in [4]. For the convenience of the reader, we recall the definition here with the dilation matrix $2I_s$. For a mask A with multiplicity m , we say that A satisfies the *sum rules* of order $k + 1$ if there exists a set of $m \times 1$ vectors $\{y_\mu : \mu \in \mathbb{N}_0^s, |\mu| \leq k\}$ with $y_0 \neq 0$ such that

$$\sum_{0 \leq \nu \leq \mu} (-1)^{|\nu|} J_\alpha^A(\nu) y_{\mu-\nu} = 2^{-|\mu|} y_\mu \quad \forall \mu \in \mathbb{N}_0^s, |\mu| \leq k, \alpha \in \mathbb{Z}^s,$$

where $(\nu_1, \dots, \nu_s) \leq (\mu_1, \dots, \mu_s)$ means $\nu_j \leq \mu_j$ for all $j = 1, \dots, s$, and

$$J_\alpha^A(\nu) := \sum_{\beta \in \mathbb{Z}^s} A(\alpha + 2\beta)(\beta + 2^{-1}\alpha)^\nu / \nu!$$

with $(\xi_1, \dots, \xi_s)^{(\nu_1, \dots, \nu_s)} := \xi_1^{\nu_1} \cdots \xi_s^{\nu_s}$. By [4, Theorem 2.4] and the remark at the end of Section 5 in [5], we see that if a mask A with multiplicity $\#M(r, s)$ satisfies the interpolatory sum rules of order $k + 1$ if and only if A satisfies the sum rules of order $k + 1$ with the vectors $\{y_\mu : \mu \in \mathbb{N}_0^s, |\mu| \leq k\}$ such that $y_\mu = e_\mu$ for $\mu \in M(r, s)$ and $y_\mu = 0$ for all $|\mu| \leq k$ and $\mu \notin M(r, s)$, where e_μ denotes the μ th unit coordinate column vector in $\mathbb{R}^{\#M}$.

Now we have the following result which is a slightly modified version of [5, Corollary 5.2]:

Theorem 5 *Denote $M = M(r, s) := \{(\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s : \mu_1 + \cdots + \mu_s \leq r\}$. Let A be a finitely supported sequence of $(\#M) \times (\#M)$ matrices on \mathbb{Z}^s . Let $\phi = (\phi_\mu)_{\mu \in M}$ be a compactly supported $1 \times \#M$ vector of generalized functions such that ϕ satisfies the refinement equation 18 with the mask A . Then ϕ is a basic Hermite interpolant of type (r, s) if and only if*

- (1) $\hat{\phi}_0(0) = 1$ (this is a normalization condition for a refinable vector function whose Fourier transform is $\hat{\phi}_0$);
- (2) $\nu_\infty(A, 2I_s) > r$, where the quantity $\nu_\infty(A, 2I_s)$ was defined in [5] and will be defined later in the Appendix for Hermite interpolatory subdivision schemes;
- (3) $A(0) = D$ and $A(2\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus \{0\}$;
- (4) The mask A satisfies the interpolatory sum rules of order $r + 1$; that is, 22 holds.

Moreover, we have $\phi \in C^{\nu_\infty(A, 2I_s) - \varepsilon}(\mathbb{R}^s)$ for any $\varepsilon > 0$.

Proof: It follows directly from [5, Corollary 5.2] and the fact that A satisfies the interpolatory sum rules of order $r + 1$ in (22) if and only if A satisfies the sum rules of order $r + 1$ with the vectors $\{y_\mu : \mu \in \mathbb{N}_0^s, |\mu| \leq r\}$ such that $y_\mu = e_\mu$ for $\mu \in M(r, s)$. The claim that $\phi \in C^{\nu_\infty(A, 2I_s) - \varepsilon}(\mathbb{R}^s)$ for any $\varepsilon > 0$ follows directly from the remarks after [5, Theorem 4.3]. \square

Motivated by the above result, as in [4,5], we say that A is a *Hermite interpolatory mask* of type (r, s) if the conditions in (3) and (4) of Theorem 5 are satisfied. The concept of Hermite interpolatory masks has been introduced in [4] in the univariate setting. In order to apply Hermite subdivision schemes with a given Hermite interpolatory mask A , symmetry on the mask A is needed. See [6] for discussion on symmetry of a Hermite interpolatory mask.

In the next section, we shall use the results in this section and in the Appendix to analyze the algorithms HD^2 and HR^2 , which are Hermite interpolatory

subdivision schemes of order 2 in dimension two and their associated masks are Hermite interpolatory masks of order 2 on \mathbb{Z}^2 .

6 Choices of the parameters and analysis of the algorithms HD^2 and HR^2

The Hermite interpolatory mask A of order 2 in either the algorithm HD^2 or HR^2 is supported on $[-1, 1]^2$. The mask A in the algorithm HD^2 is given in (7), (8) and (9) and the mask A in the algorithm HR^2 is given in (7), (8) and (10). When

$$\begin{aligned} \lambda_{00} &= 1/2, & \lambda_{01} &= -1/8, & \lambda_{10} &= 1 - 2\lambda_{11}, \\ \kappa_{00} &= 1/2, & \kappa_{01} &= -2\kappa_{02} - 1/8, & \kappa_{10} &= 3/2 + 12\kappa_{12}, \\ \kappa_{11} &= -6\kappa_{12} - 1/4, & \kappa_{20} &= 0, & \kappa_{21} &= 1 - 2\kappa_{22}, \end{aligned} \quad (24)$$

the mask A in either the algorithm HR^2 or algorithm HD^2 satisfies the interpolatory sum rules of order 4. In the following discussion of both algorithms HD^2 and HR^2 , we assume that (24) are satisfied.

Using the method described in Section 6 and the technique described in the Appendix, for the algorithm HD^2 , we have the following proposition about the Hölder smoothness exponent $\nu_\infty(A, 2I_s)$. This exponent is described in the Appendix and the proof of the proposition is given at the same place.

Proposition 6 *For the algorithm HD^2 , we set $\lambda_{11} = -11/32$, $\kappa_{02} = 5/256$, $\kappa_{12} = 1/16$, $\kappa_{22} = -13/32$ and we assume that (24) holds. Then we have $\nu_\infty(A, 2I_2) \geq 2.8363$. So, the algorithm HD^2 yields a C^2 Hermite interpolant and its Hermite interpolatory subdivision scheme of order 2 converges in C^2 .*

Similarly, using the method described in Section 6 and the technique described in the Appendix, for the algorithm HR^2 , we have

Proposition 7 *For the algorithm HR^2 , we set $\lambda_{11} = -15/64$, $\kappa_{02} = 1/64$, $\kappa_{12} = 1/64$, $\kappa_{22} = -15/64$ and we assume that (24) holds. Then we have $\nu_\infty(A, 2I_2) \geq 2.9175$. So, the algorithm HR^2 yields a C^2 Hermite interpolant and its Hermite interpolatory subdivision scheme of order 2 converges in C^2 .*

From a given C^2 Hermite subdivision scheme \mathcal{H} of type (2,2), many graphs of surfaces can be drawn. For the six basic Hermite interpolants on the square $[-1, 1]^2, \phi_k, k = 1, 2, \dots, 6$ and their partial derivatives of orders 1 and 2, we get 36 surfaces. We choose to draw some of these for $\mathcal{H} = HR^2$ with the choice of parameters $\lambda_{11} = -15/64, \kappa_{02} = 1/64, \kappa_{12} = 1/64, \kappa_{22} = -15/64$ (as in Proposition 7). In Figure 3, we draw the graph of the first basic Hermite interpolant $z = \phi_1(x, y)$ and the graphs of three partial derivatives of ϕ_1 . In

Figure 4, we draw the graphs of the partial derivative $\partial^2/\partial x^2$ of the two basic Hermite interpolants ϕ_3, ϕ_4 and their contour lines. In Figure 5, we draw the graph of the mixed derivative of order 2 of ϕ_5 . From the last Figure, we see that \mathcal{H} is not C^3 , there is no tangent plane at the origin.

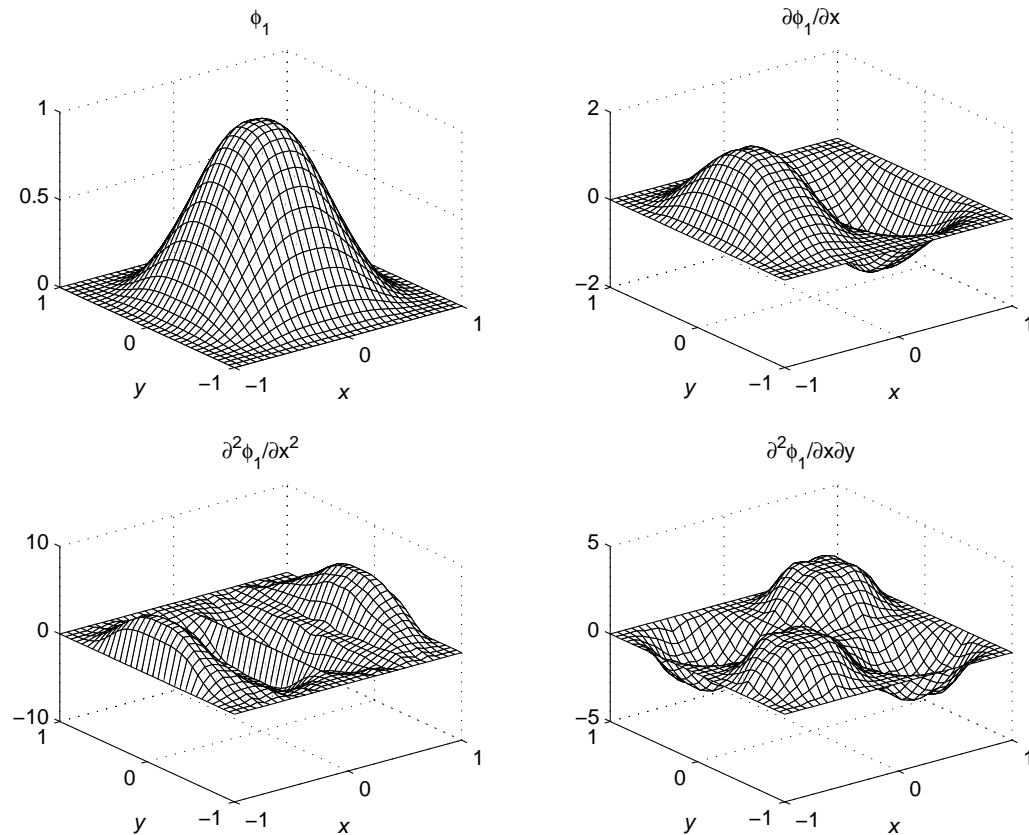


Fig. 3. Graphs of the first Hermite interpolant ϕ_1 and of its partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial x \partial y}$.

Theorem 8 *Let the Hermite interpolatory subdivision scheme $\mathcal{H} = HR^2$ as described in Proposition 7. Let $f_n : \mathbb{Z}^2 \rightarrow \mathbb{R}^6$ be a sequence of refinements of \mathcal{H} . There exists a C^2 -function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that*

$$f_n^\mu(\alpha) = \partial^\mu \phi(\alpha/2^n)$$

for any 2-tuple μ such that $|\mu| \leq 2$ and for any $\alpha \in \mathbb{Z}^2$. Let $x \in [i, i+1] \times [j, j+1]$, we set

$$m_k = \sum_{v \in \{i, i+1\} \times \{j+1\}} |f_0^{(k)}(v)|, k = 1, 2, \dots, 6$$

where $f_0^{(k)}$ is the k th-component of f_0 . Then

$$|\partial_k^\mu \phi(x)| \leq \sum_{\ell=1}^6 b_{k\ell} m_\ell \quad (25)$$

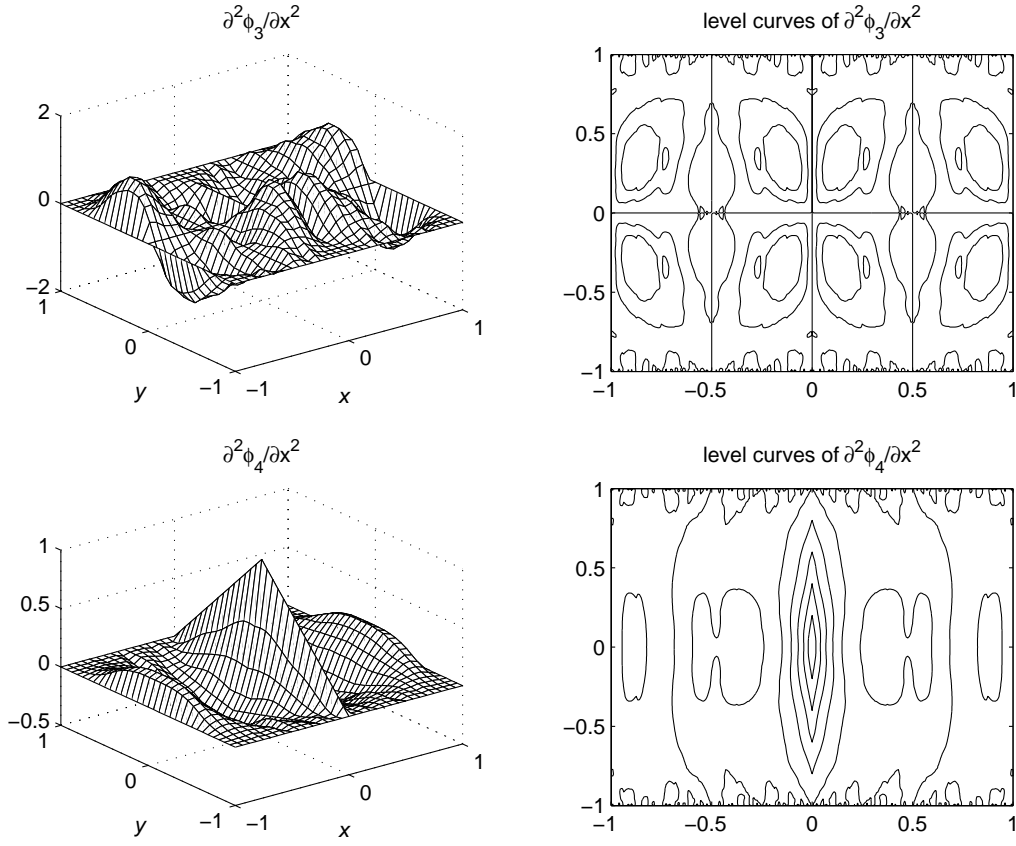


Fig. 4. Graphs and contour lines of the partial derivative $\partial^2/\partial x^2$ of two basic Hermite interpolants.

where $\mu_1 = (0, 0)$, $\mu_2 = (1, 0)$, $\mu_3 = (0, 1)$, $\mu_4 = (2, 0)$, $\mu_5 = (1, 1)$, $\mu_6 = (0, 2)$ and where $(b_{k\ell})$ is the following matrix

$$\begin{pmatrix} 1.000 & 0.189 & 0.189 & 0.016 & 0.028 & 0.016 \\ 1.691 & 1.000 & 0.297 & 0.061 & 0.147 & 0.027 \\ 1.691 & 0.297 & 1.000 & 0.027 & 0.147 & 0.061 \\ 5.666 & 3.712 & 1.189 & 1.000 & 0.557 & 0.100 \\ 2.489 & 1.474 & 1.474 & 0.095 & 1.000 & 0.095 \\ 5.666 & 1.189 & 3.712 & 0.100 & 0.557 & 1.000 \end{pmatrix}.$$

Proof: Let $\Phi = (\phi_{k\ell})$ be the basic matrix function of \mathcal{H} . If $x \in [i, i+1] \times [j, j+1]$, then

$$\partial^{\mu_k} \phi(x) = \sum_{\alpha \in \{i, i+1\} \times \{j, j+1\}} \sum_{\ell=1}^6 \phi_{k\ell}(x - \alpha) f_0^{(\ell)}(\alpha).$$

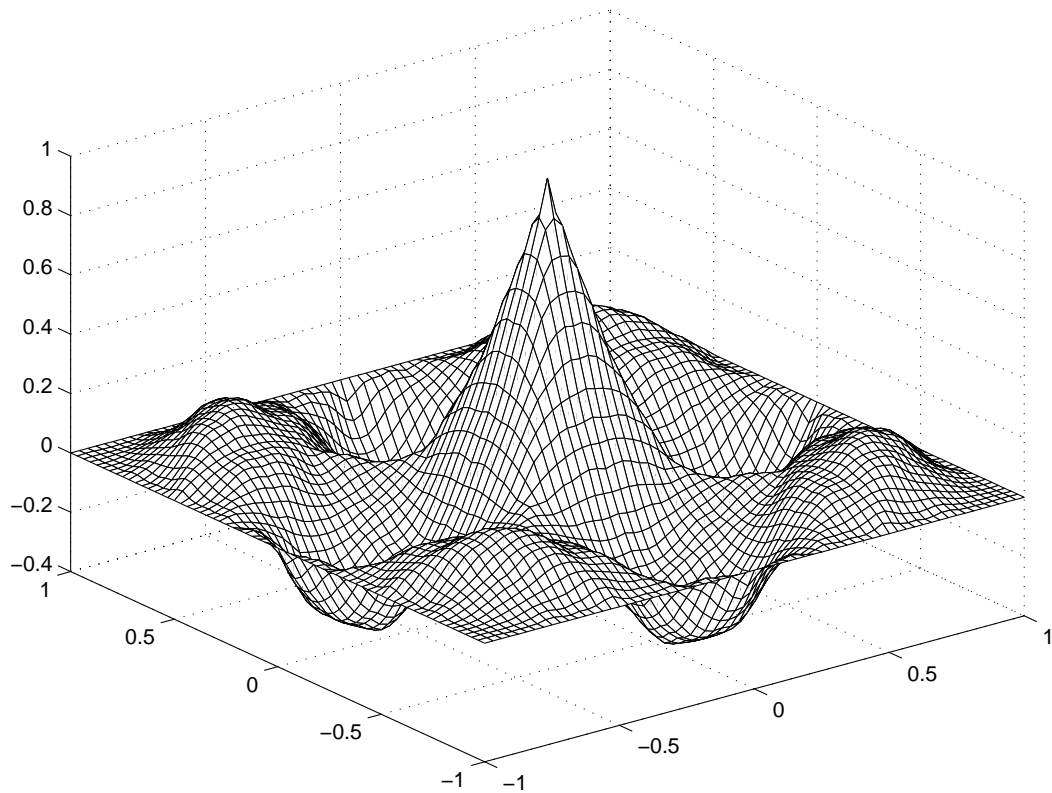


Fig. 5. Graph of the mixed second partial derivative of the 5th basic Hermite interpolant.

We get

$$|\partial^{\mu_k} \phi(x)| \leq \sum_{\ell=1}^6 b_{k\ell} m_\ell$$

where $b_{k\ell} = \max\{|\phi_{k\ell}(x)| : x \in [-1, 1]^2\}$. The numbers $b_{k\ell}$ are found after the computation of the matrix function $\Phi_n : \mathbb{Z}^2 \rightarrow \mathbb{R}^{6 \times 6}$ at the refinement level $n = 10$. \square

7 Interpolation on a rectangular mesh

We describe the construction of a C^2 interpolant on a semiregular rectangular mesh. We suppose that the data f and its first and second derivatives are given on a grid: $\{x_0, x_1, \dots, x_n\} \times \{y_0, y_1, \dots, y_m\}$. Let (x_i, y_j) be the southwest vertex of an elementary rectangle and let $h_i = x_{i+1} - x_i$, $k_j = y_{j+1} - y_j$. On the vertices of $[0, 1]^2$, we define the initial data of a function $g : \{0, 1\}^2 \rightarrow \mathbb{R}$

by

$$\begin{aligned}
g(u, v) &= f(x_i + uh_i, y_j + vk_i), \\
g_x(u, v) &= p(u, v) = h_i f_x(x_i + uh_i, y_j + vk_i), \\
g_y(u, v) &= q(u, v) = k_i f_y(x_i + uh_i, y_j + vk_i), \\
g_{x^2}(u, v) &= r(u, v) = h_i^2 f_{x^2}(x_i + uh_i, y_j + vk_i), \\
g_{xy}(u, v) &= s(u, v) = h_i k_i f_{xy}(x_i + uh_i, y_j + vk_i), \\
g_{y^2}(u, v) &= t(u, v) = k_i^2 f_{y^2}(x_i + uh_i, y_j + vk_i).
\end{aligned} \tag{26}$$

Now at each step n , for $(u, v) \in \{0, 1/2^n, 2/2^n, \dots, 1\}^2$, we build $g(u, v)$ and its derivatives by HD^2 or HR^2 . If the scheme is convergent, we can extend g in a C^2 function on the square. The reverse formulae of (26) give an interpolating C^2 function f with its derivatives on $\{x_i, x_{i+1}\} \times \{y_j, y_{j+1}\}$

In order to get C^2 continuity on the whole domain, it suffices to study it when crossing an edge. Formulae 4, 5 and 6 give the construction of g and its derivatives on the edge $\{0, 1\} \times \{0\}$. This construction is done through schemes in dimension 1 and the computed values of f and its derivatives are independent of the second dimension and independent of the two other vertices of the elementary rectangle.

For example, at step 1, at $(x_i + h_i/2, y_j)$ i.e. $(u, v) = (1/2, 0)$, $g(1/2, 0) = \kappa_{00}[g(1, 0) + g(0, 0)] + \kappa_{01}[p(1, 0) - p(0, 0)] + \kappa_{02}[r(1, 0) + r(0, 0)]$ gives $f(x_i + h_i/2, y_j) = \kappa_{00}[f(x_i + h_i, y_j) + f(x_i, y_j)] + \kappa_{01}[h_i f_x(x_i + 1, y_j) - h_i f_x(x_i, y_j)] + \kappa_{02}[h_i^2 f_{x^2}(x_i + 1, y_j) + h_i^2 f_{x^2}(x_i, y_j)]$ and $q(1/2, 0) = \lambda_{00}[q(1, 0) + q(0, 0)] + \lambda_{01}[s(1, 0) - s(0, 0)]$ gives $k_i f_y(x_i + h_i/2) = \lambda_{00}[k_i f_y(x_i + h_i, y_j) + k_i f_y(x_i, y_j)] + \kappa_{01}[h_i k_i f_{xy}(x_i + 1, y_j) - h_i k_i f_{xy}(x_i, y_j)]$ then $f_y(x_i + h_i/2) = \lambda_{00}[f_y(x_i + h_i, y_j) + f_y(x_i, y_j)] + \kappa_{01}[h_i f_{xy}(x_i + 1, y_j) - h_i f_{xy}(x_i, y_j)]$. We have similar results for the other formulae of Case 1 and at each step. As soon as f is C^2 on $\{x_i, x_{i+1}\} \times \{y_j, y_{j+1}\}$ and on $\{x_i, x_{i+1}\} \times \{y_{j-1}, y_j\}$, then f is C^2 on $\{x_i, x_{i+1}\} \times \{y_{j-1}, y_{j+1}\}$. We have a similar result in the other direction.

For Figure 6, we have interpolated the function $z(x, y) = \frac{1}{1 + x^4 + y^2}$ and its derivatives on the vertices of the mesh $\{-3, -2, -1, 0, 2\} \times \{-3, -1.5, 0, 1, 3\}$ by HR^2 with the mask given in Proposition 7 and (24). We stopped the construction at step $n = 5$.

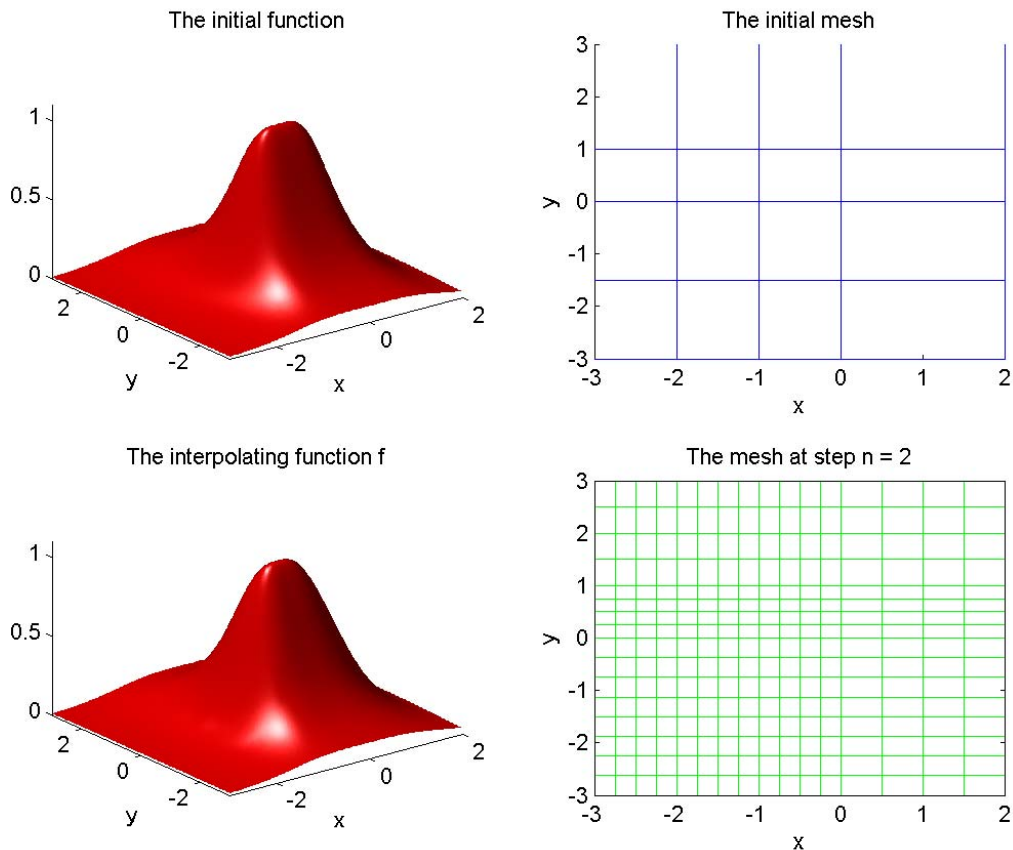


Fig. 6. Interpolating function on a rectangular mesh.

8 Conclusion

We were able to produce a Hermite subdivision scheme of type $(2, 2)$ with mask A such that $\nu_\infty(A, 2I_2) \geq 2.9175$. But for this mask A we do not know the precise value of $\nu_\infty(A, 2I_2)$. An open question is the following. Can one find a Hermite subdivision scheme \mathcal{H} of type $(2, 2)$ whose support is contained in $[-1, 1]^2$ and such that $\nu_\infty(A, 2I_2) = 3$ (in order to get $C^{3-\epsilon}$ interpolant $\forall \epsilon > 0$)?

We saw that both algorithms HD^2 and HR^2 give efficient tools for the construction of C^2 interpolating surfaces on a semiregular rectangular mesh. Unlike the tensor product method, notice that we do not use derivatives of order 3 to get C^2 . Nevertheless, if one or more derivatives of order ≤ 2 are unknown, they can be approximated by finite differences.

The algorithms give a method to build a surface $z = f(x, y)$, $(x, y) \in [0, 1]^2$ or more generally C^2 parametric surfaces $S(u, v) = (x(u, v), y(u, v), z(u, v))$, since the computations can be done component by component. An example of a construction of a sphere is given in Figure 7. We have interpolated $x(u, v) =$

$\cos(u)\cos(v)$, $y(u, v) = \sin(u)\cos(v)$, $z(u, v) = \sin(v)$ at the 4 vertices of the square $[0, \pi/2]^2$ then we have completed with symetries.

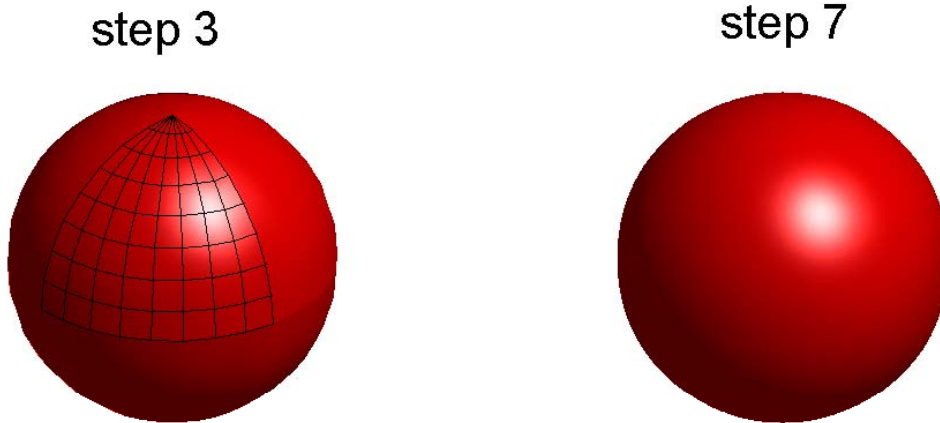


Fig. 7. Interpolation of a sphere.

We conclude with a very pertinent comment from one of the referees. “Another issue related to geometric design is how to model with C^2 data. Such schemes provide additional degrees of freedom which can be modified by the designer, the major problem is how to intuitively modify parameters like tangent planes and curvature (using finite difference schemes on a control mesh given by the user does not explore the newly created parameter space to the full extent).” This is another open question that deserves further investigation.

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions. The first author was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Canada). The research of the second author was supported in part by NSERC Canada under Grant G121210654.

Appendix

In this Appendix, we shall recall the definition of the important quantity $\nu_p(A, 2I_s)$, $1 \leq p \leq \infty$, for the Hermite interpolatory subdivision schemes. Then for the two algorithms HD^2 and HR^2 , we shall discuss how to estimate the Hölder smoothness exponent $\nu_\infty(A, 2I_s)$.

Let us recall the definition of $\nu_p(A, 2I_s)$ in [5] for the special case of Hermite interpolatory masks. Throughout this Appendix, A denotes a finitely supported matrix mask and the dilation matrix is $2I_s$.

The convolution of two sequences is defined to be

$$[u * v](\alpha) := \sum_{\beta \in \mathbb{Z}^s} u(\beta)v(\alpha - \beta), \quad u \in (\ell_0(\mathbb{Z}^s))^{m \times n}, v \in (\ell_0(\mathbb{Z}^s))^{n \times j}.$$

As in [5], we define a space $\mathcal{V}_{k,r}$ by

$$\mathcal{V}_{k,r} := \{v \in (\ell_0(\mathbb{Z}^s))^{1 \times \#M} : [v * p](0) = \sum_{\beta \in \mathbb{Z}^s} v(-\beta)p(\beta) = 0 \quad \forall p \in \mathcal{P}_{k,r}\} \quad (\text{A.1})$$

where $\mathcal{P}_{k,r}$ is defined in (19), and

$$\rho_k(A, 2I_s, p) := \sup\{\|v * [S_A^n(\delta I_{\#M(r,s)})]\|_{(\ell_p(\mathbb{Z}^s))^{1 \times \#M}}^{1/n} : v \in \mathcal{V}_{k,r}\}. \quad (\text{A.2})$$

We say that a subset \mathcal{B} of $\mathcal{V}_{k,r}$ generates $\mathcal{V}_{k,r}$ if $\text{span}\{v(\cdot - \alpha) : v \in \mathcal{B}, \alpha \in \mathbb{Z}^s\} = \mathcal{V}_{k,r}$. If \mathcal{B} generates $\mathcal{V}_{k,r}$, then it is easy to see that

$$\rho_p(A, 2I_s, p) = \sup\{\|v * [S_A^n(\delta I_{\#M(r,s)})]\|_{(\ell_p(\mathbb{Z}^s))^{1 \times \#M}}^{1/n} : v \in \mathcal{B}\}.$$

Now we define the important quantity $\nu_p(A, 2I_s)$ for a Hermite interpolatory mask A of type (r, s) as follows: Let k be the largest integer such that the mask A satisfies the interpolatory sum rules of order $k + 1$. Define

$$\nu_p(A, 2I_s) := s/p - \log_2 \rho_k(A, 2I_s, p), \quad 1 \leq p \leq \infty. \quad (\text{A.3})$$

See [5] for a more general definition of the quantity $\nu_p(A, 2I_s)$, which plays a very important role in the convergence and smoothness analysis of vector subdivision schemes (see [5]). In general, it is difficult to calculate the exact Hölder smoothness exponent $\nu_\infty(A, 2I_s)$, even for dimension 1, see [2,5,6,10] and references therein for detail.

The quantity $\rho_k(A, 2I_s, p)$ defined in (A.2) can be equivalently rewritten using the ℓ_p -norm joint spectral radius. Let \mathcal{T} be a finite collection of linear operators acting on a finite-dimensional normed vector space V . For a positive integer n , \mathcal{T}^n denotes $\mathcal{T}^n = \{(T_1, \dots, T_n) : T_1, \dots, T_n \in \mathcal{T}\}$, and for $1 \leq p \leq \infty$, we define

$$\|\mathcal{T}^n\|_p^p := \sum_{(T_1, \dots, T_n) \in \mathcal{T}^n} \|T_1 \cdots T_n\|^p$$

and

$$\|\mathcal{T}^n\|_\infty := \max\{\|T_1 \cdots T_n\| : (T_1, \dots, T_n) \in \mathcal{T}^n\},$$

where $\|\cdot\|$ denotes an operator norm. For $1 \leq p \leq \infty$, the ℓ_p -norm joint spectral radius of \mathcal{T} (see [3,11] and references therein) is defined to be

$$\rho_p(\mathcal{T}) := \lim_{n \rightarrow \infty} \|\mathcal{T}^n\|_p^{1/n} = \inf_{n \geq 1} \|\mathcal{T}^n\|_p^{1/n}. \quad (\text{A.4})$$

Let Γ denote the set of all the vertices of the cube $[0, 1]^s$. To relate the quantity $\rho_k(A, 2I_s, p)$ with the ℓ_p -norm joint spectral radius, we introduce the linear operators $T_\gamma (\gamma \in \Gamma)$ on $(\ell_0(\mathbb{Z}^s))^{1 \times \#M}$ by

$$T_\gamma v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} v(\beta) A(2\alpha - \beta + \gamma), \quad \alpha \in \mathbb{Z}^s, v \in (\ell_0(\mathbb{Z}^s))^{1 \times \#M}. \quad (\text{A.5})$$

It was proved in [3, Lemma 2.3] that if A is finitely supported, then for any finitely supported sequence v on \mathbb{Z}^s , there exists a finite dimensional subspace $V(v)$ of $(\ell_0(\mathbb{Z}^s))^{1 \times \#M}$ such that $V(v)$ contains v and $V(v)$ is the smallest subspace of $(\ell_0(\mathbb{Z}^s))^{1 \times \#M}$ which is invariant under all the operators $T_\gamma, \gamma \in \Gamma$. We call such $V(v)$ the minimal $\{T_\gamma : \gamma \in \Gamma\}$ invariant subspace generated by v .

Let $\mathcal{T} := \{T_\gamma|_W : \gamma \in \Gamma\}$ where W is the minimal $\{T_\gamma : \gamma \in \Gamma\}$ invariant subspace generated by a finite subset \mathcal{B} of $\mathcal{V}_{k,r}$ such that \mathcal{B} generates the space $\mathcal{V}_{k,r}$. It is known from [3] that

$$\rho_k(A, 2I_s, p) = \rho_p(\mathcal{T}). \quad (\text{A.6})$$

It was shown in [11] that when p is an even integer, one can compute $\rho_p(\mathcal{T})$ by calculating the spectral radius of a certain finite matrix. Let us recall this result from [11] here.

By $B \otimes C$ we denote the (right) Kronecker product of two matrices. In particular, $\otimes^k B := B \otimes \cdots \otimes B$ with k -copies of B . Let B_γ denote the matrix representation of the operator T_γ in (A.5) under a given basis of W . It was shown in [11] that

$$\rho_{2k}(\mathcal{T}) = [\rho(\mathcal{M})]^{1/2k} \quad \text{with} \quad \mathcal{M} := \sum_{\gamma \in \Gamma} \otimes^k [\overline{B_\gamma} \otimes B_\gamma]. \quad (\text{A.7})$$

If the dimension of the space W is n , then \mathcal{M} is an $n^{2k} \times n^{2k}$ matrix. Therefore, in order to make the above calculation feasible, it is crucial for us to find a subset \mathcal{B} of $\mathcal{V}_{k,r}$ such that the dimension of W is as small as possible.

Let us employ the technique described above to analyze the algorithms HD^2 and HR^2 . In the following discussion of both algorithms HD^2 and HR^2 , we assume that (24) are satisfied. Therefore, the mask A in either the algorithm HR^2 or algorithm HD^2 satisfies the interpolatory sum rules of order 4.

Lemma 9 *Let $\Gamma := \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and $K_0 := -\Gamma = \{(0, 0), (-1, 0), (0, -1), (-1, -1)\}$. Denote $W_0 = (\ell(K_0))^{1 \times 6} \cap \mathcal{V}_{2,2}$, where $\ell(K_0)$ denotes all the sequences which vanish outside K_0 . Then W_0 is invariant under all the operators $T_\gamma, \gamma \in \Gamma$ defined in (A.5) with the mask A in either HD^2 or HR^2 . Moreover, W_0 generates $\mathcal{V}_{2,2}$; that is,*

$$\text{span}\{v(\cdot - \beta) : v \in W_0, \beta \in \mathbb{Z}^2\} = \mathcal{V}_{2,2}$$

and $\dim(W_0) = 18$.

Proof: Since A satisfies the interpolatory sum rules of order 3, we have $S_A \mathcal{P}_{2,2} \subseteq \mathcal{P}_{2,2}$ and we know that $\mathcal{V}_{2,2}$ is invariant under all the operators T_γ ($\gamma \in \Gamma$). By a simple argument (see [3]), we observe that $(\ell(K_0))^{1 \times 6}$ is invariant under all the operators T_γ since $2K_0 \subseteq K_0 + [-1, 1]^2 \cap \mathbb{Z}^2 - \Gamma$. So, the space W_0 is invariant under all the operators T_γ ($\gamma \in \Gamma$).

We show that for any $\alpha \in \mathbb{Z}^2$, $(\ell(\{\alpha\}))^{1 \times 6} \cap \mathcal{V}_{2,2} = \{0\}$. The linear space $\mathcal{P}_{2,2}$ has a basis $\{p_1, p_2, p_3, p_4, p_5, p_6\}$, where

$$\begin{aligned} p_1 &:= [1, 0, 0, 0, 0, 0]^T, & p_2 &:= [x, 1, 0, 0, 0, 0]^T, & p_3 &:= [y, 0, 1, 0, 0, 0]^T, \\ p_4 &:= [x^2, 2x, 0, 2, 0, 0]^T, & p_5 &:= [xy, y, x, 0, 1, 0]^T, & p_6 &:= [y^2, 0, 2y, 0, 0, 2]^T. \end{aligned}$$

The linear space $(\ell(\{\alpha\}))^{1 \times 6}$ has a basis $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, where $v_j = \delta(\cdot - \alpha)e_j$, $j = 1, \dots, 6$, where e_j denotes the j th unit coordinate row vector in \mathbb{R}^6 . Note that $\sum_{\beta \in \mathbb{Z}^2} v_j(-\beta)p_i(\beta) = p_i(-\alpha)e_j$. Therefore,

$$\left(\sum_{\beta \in \mathbb{Z}^2} v_j(-\beta)p_i(\beta) \right)_{1 \leq i, j \leq 6} = \left(p_i(-\alpha)e_j \right)_{1 \leq i, j \leq 6}$$

is a lower triangular matrix and it is of full rank. Therefore, by the definition of $\mathcal{V}_{2,2}$ in (A.1), we conclude that $(\ell(\{\alpha\}))^{1 \times 6} \cap \mathcal{V}_{2,2} = \{0\}$. Consequently, we conclude that for any finite nonempty subset K of \mathbb{Z}^2 ,

$$\dim((\ell(K))^{1 \times 6} \cap \mathcal{V}_{2,2}) = 6(\#K) - 6.$$

Since for any $\alpha \in \mathbb{Z}^2$, $\{v(\cdot - \alpha) : v \in \mathcal{V}_{2,2}\} = \mathcal{V}_{2,2}$ (see [5, Proposition 2.5]), it is easy to see that $\{v(\cdot - \alpha) : v \in (\ell(K))^{1 \times 6} \cap \mathcal{V}_{2,2}\} = (\ell(K + \alpha))^{1 \times 6} \cap \mathcal{V}_{2,2}$.

Let K_1 and K_2 be two finite subsets of \mathbb{Z}^2 such that $K_1 \cap K_2 \neq \emptyset$. We have

$$\begin{aligned} \dim((\ell(K_1 \cup K_2))^{1 \times 6} \cap \mathcal{V}_{2,2}) &+ \dim((\ell(K_1 \cap K_2))^{1 \times 6} \cap \mathcal{V}_{2,2}) \\ &= 6(\#(K_1 \cup K_2)) + 6(\#(K_1 \cap K_2)) - 12 \\ &= 6(\#K_1) + 6(\#K_2) - 12 \\ &= \dim((\ell(K_1))^{1 \times 6} \cap \mathcal{V}_{2,2}) + \dim((\ell(K_2))^{1 \times 6} \cap \mathcal{V}_{2,2}). \end{aligned}$$

It is well known that for two subspaces U_1 and U_2 , one has

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2), \quad (\text{A.8})$$

Consequently, by the fact

$$(\ell(K_1))^{1 \times 6} \cap \mathcal{V}_{2,2} + (\ell(K_2))^{1 \times 6} \cap \mathcal{V}_{2,2} \subseteq (\ell(K_1 \cup K_2))^{1 \times 6} \cap \mathcal{V}_{2,2}$$

and (A.8), we deduce that

$$(\ell(K_1))^{1 \times 6} \cap \mathcal{V}_{2,2} + (\ell(K_2))^{1 \times 6} \cap \mathcal{V}_{2,2} = (\ell(K_1 \cup K_2))^{1 \times 6} \cap \mathcal{V}_{2,2}.$$

Now it is straightforward to see that $(\ell(K_0))^{1 \times 6} \cap \mathcal{V}_{2,2}$ generates $\mathcal{V}_{2,2}$ since $\mathbb{Z}^2 = \cup_{\alpha \in \mathbb{Z}^2} (K_0 + \alpha)$. \square

Since A satisfies the interpolatory sum rules of order 4, we have $S_A \mathcal{P}_{3,2} \subseteq \mathcal{P}_{3,2}$ and we know that $W_1 := (\ell(K_0))^{1 \times 6} \cap \mathcal{V}_{3,2}$ is invariant under all the operators T_γ ($\gamma \in \Gamma$), where K_0 is given in Lemma 9. Therefore, W_1 is an invariant subspace of W_0 . Moreover, by computation, we see that $\dim(W_0) - \dim(W_1) = 4$. Let

$$\begin{aligned} u_1 &= 2\delta_{(0,0)}e_1 + \delta_{(0,0)}e_2 - 2\delta_{(-1,0)}e_1 + \delta_{(-1,0)}e_2, \\ u_2 &= \delta_{(0,0)}e_3 - \delta_{(-1,0)}e_3 + \delta_{(-1,0)}e_5, \\ u_3 &= \delta_{(0,0)}e_2 - \delta_{(0,-1)}e_2 + \delta_{(0,-1)}e_5, \\ u_4 &= 2\delta_{(0,0)}e_1 + \delta_{(0,0)}e_3 - 2\delta_{(0,-1)}e_1 + \delta_{(0,-1)}e_3 \end{aligned}$$

and

$$\begin{aligned} p_7 &:= [x^3, 3x^2, 0, 6x, 0, 0]^T, & p_8 &:= [x^2y, 2xy, x^2, 2y, 2x, 0]^T, \\ p_9 &:= [xy^2, y^2, 2xy, 0, 2y, 2x]^T, & p_{10} &:= [y^3, 0, 3y^2, 0, 0, 6y]^T, \end{aligned}$$

where $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^2 \setminus \{\alpha\}$, and e_j denotes the j th coordinate unit row vector in \mathbb{R}^6 . Then all $u_j \in W_0$ and $\sum_{\beta \in \mathbb{Z}^2} p_i(\beta)u_j(-\beta) = \delta(i - j - 6)$ for all $i = 7, 8, 9, 10$ and $j = 1, 2, 3, 4$. Let $[u_j]$ denote the element represented by u_j in the quotient group W_0/W_1 . Then $\{[u_j] : j = 1, \dots, 4\}$ is a basis for the quotient group W_0/W_1 . Moreover, the representation matrix of A_γ under this basis is $\text{diag}(1/8, 1/8, 1/8, 1/8)$ for every $\gamma \in \Gamma$. Therefore,

$$\begin{aligned} \rho_p(\{T_\gamma|_{W_0} : \gamma \in \Gamma\}) &= \max\{\rho_p(T_\gamma|_{W_0/W_1} : \gamma \in \Gamma), \rho_p(\{T_\gamma|_{W_1} : \gamma \in \Gamma\})\} \\ &= \max\{2^{2/p-3}, \rho_p(\{T_\gamma|_{W_1} : \gamma \in \Gamma\})\}. \end{aligned} \quad (\text{A.9})$$

Denote

$$\gamma_p(A) := s/p - \log_2 \rho_p(\{T_\gamma|_{W_1} : \gamma \in \Gamma\}), \quad 1 \leq p \leq \infty. \quad (\text{A.10})$$

It is known (see [3] or [5, (4.8)]) that $\gamma_\infty(A) \geq \gamma_p(A) - 2/p$ for all $1 \leq p \leq \infty$. Moreover, it follows from (A.9) that

$$\nu_\infty(A, 2I_2) \geq \min\{3, \gamma_\infty(A)\}.$$

In order to find a C^2 Hermite interpolant from either the algorithm HD^2 or HR^2 , we first try to find some choices of the parameters $\lambda_{11}, \kappa_{02}, \kappa_{12}, \kappa_{22}$ such that the quantity $\gamma_2(A)$ is relatively large. Note that $\dim(W_1) = 14$. When $k = 2$, the matrix \mathcal{M} in (A.7) is a $14^4 \times 14^4$ matrix which is too large for us to compute its spectral radius. For an $m \times n$ matrix $v = (v_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, denote

$$\text{vec}(v) := (v_{1,1}, \dots, v_{m,1}, v_{1,2}, \dots, v_{m,2}, \dots, v_{n,1}, \dots, v_{m,n})^T.$$

It is well known that $\text{vec}(AvB) = (B^T \otimes A)\text{vec}(v)$. Let $n := \dim(W)$. So, $B_\gamma, \gamma \in \Gamma$ are $n \times n$ matrices. Define a linear transform:

$$N : (\ell_0(\mathbb{Z}^2))^{n \times n} \mapsto (\ell_0(\mathbb{Z}^2))^{n \times n}, \quad N(v) = \sum_{\gamma \in \Gamma} B_\gamma v \overline{B_\gamma}^T, \quad v \in (\ell_0(\mathbb{Z}^2))^{n \times n}.$$

By the relation $\text{vec}(N(v)) = \sum_{\gamma \in \Gamma} (\overline{B_\gamma} \otimes B_\gamma)\text{vec}(v)$, we see that the matrix representation of the linear transform N is $\mathcal{M} = \sum_{\gamma \in \Gamma} \overline{B_\gamma} \otimes B_\gamma$. Now we can use Maple to find a basis for the space W_1 and use the MATLAB routine `eigs` (with the option to deal with “sparse” linear transforms) to compute the largest eigenvalue (in modulus) of N to get the spectral radius of \mathcal{M} .

Using the method described in Section 3 and the technique described above, for the algorithm HD^2 , we set $\lambda_{11} = -11/32, \kappa_{02} = 5/256, \kappa_{12} = 1/16, \kappa_{22} = -13/32$ and assume that (24) holds. Then $\gamma_2(A) \approx 3.3777$ and $\gamma_4(A) \approx 3.3363$. Consequently, $\gamma_\infty(A) \geq \gamma_4(A) - 1/2 \approx 2.8363$ and therefore, by $\nu_\infty(A, 2I_2) \geq \min\{3, \gamma_\infty(A)\}$, we have $\nu_\infty(A, 2I_2) \geq 2.8363$. So, the algorithm HD^2 yields a C^2 Hermite interpolant and its Hermite interpolatory subdivision scheme of order 2 converges in C^2 . Therefore, we have Proposition 6.

Similarly, using the method described in Section 3 and the technique described above, for the algorithm HR^2 , we set $\lambda_{11} = -15/64, \kappa_{02} = 1/64, \kappa_{12} = 1/64, \kappa_{22} = -15/64$ and assume that (24) holds. Then $\gamma_2(A) \approx 3.5596$ and $\gamma_4(A) \approx 3.4175$. Consequently, $\gamma_\infty(A) \geq \gamma_4(A) - 1/2 \approx 2.9175$ and therefore, by $\nu_\infty(A, 2I_2) \geq \min\{3, \gamma_\infty(A)\}$, we have $\nu_\infty(A, 2I_2) \geq 2.9175$. So, the algorithm HR^2 yields a C^2 Hermite interpolant and its Hermite interpolatory subdivision scheme of order 2 converges in C^2 . Therefore, we have Proposition 7.

Remark 2 We compare with another computation. We set $\Lambda = \begin{pmatrix} 1/2 & -1/8 \\ 3/2 & -1/4 \end{pmatrix}$

and $K = \begin{pmatrix} 1/2 & -5/32 & 1/64 \\ 15/8 & -7/16 & 1/32 \\ 0 & 3/2 & -1/4 \end{pmatrix}$ and we apply both algorithms HD^2 and HR^2

with these 2 matrix parameters. The masks A_1 and A_2 of these algorithms are given by Formulae (7-8-9) and (7-8-10) respectively. When we apply the HD^2 algorithm, we get $\gamma_4(A_1) \approx 3.1797$ and $\nu_\infty(A_1, 2I_2) \geq 2.6797$. When we apply the HR^2 algorithm, we get $\gamma_4(A_2) \approx 3.3270$ and $\nu_\infty(A_2, 2I_2) \geq 2.8270$.

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