Convergent Vector and Hermite Subdivision Schemes

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Abstract

Hermite subdivision schemes have been studied by Merrien, Dyn and Levin and they appear to be very different from subdivision schemes analyzed before since the rules depend on the subdivision level. As suggested by Dyn and Levin, it is possible to transform the initial scheme into a uniform stationary vector subdivision scheme which can be handled more easily. With this transformation, the study of convergence of Hermite subdivision schemes is reduced to that of vector stationary subdivision schemes. We propose a first criterion for C^0 convergence for a large class of vector subdivision schemes. This gives a criterion for C^1 convergence of Hermite subdivision schemes. It can be noticed that these schemes do not have to be interpolatory. We conclude by investigating spectral properties of Hermite schemes and other necessary/sufficient conditions of convergence.

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1 Introduction

Subdivision methods constitute a large class of recursive schemes for computing curves in $\mathbb{R}^r$, see Cavaretta, Dahmen and Michelli [1]. Vector subdivision schemes, matrix refinement equations and their links with the wavelets in multidimension have been studied by many authors, for example Cohen et al. [2, 3], Daubechies and Lagarias [4], Heil and Colella [13], Jia et al. [14]. They usually give the regularity of the solution of a wavelet equation in distribution or Sobolev spaces. Moreover deep studies of Hermite subdivision schemes have been done by Dyn and Levin [8, 9], Zhou [18], Han [10, 11] and Han et al. [12].

The main purpose of this paper is to investigate the C^0-convergence of a refinement scheme of dimension d with a matrix mask. We are especially interested in solutions which converge to continuous vector functions $\Phi = (\phi_1, ..., \phi_d)^T$ such that

$\phi_k = \phi_1, k = 1, \dots, d$. First we propose a criterion for C^0 convergence for a large class of vector subdivision schemes and then we apply this criterion to Hermite subdivision schemes. Dyn and Levin [8, 9] have given a condition for C^1-convergence of interpolating Hermite schemes. With our approach, we are able to generalize their result to arbitrary non interpolatory schemes and our criterion is much more explicit.

Our paper can be detailed as follows. In Section 2, we introduce the vector subdivision scheme $g_{n+1}(i) = \sum_{j\in\mathbb{Z}} B_{i-2j} g_n(j)$ where $g_n(i) \in \mathbb{R}^d$ and B_i is a mask of $d \times d$ matrices. Section 3 is devoted to the definition of C^0-convergence and to the first properties of the vector schemes. In Section 4, we give sufficient conditions to obtain the C^0-convergence. The two conditions are the affinity of the scheme which is a usual condition and another one based on the n-th norming factor κ_n.

Then, in Section 5 we define the Hermite subdivision schemes $\mathcal{H}$ in order to get a function and its first derivative. Since we want to get a solution of the type $f = (\phi, \phi')^T$, the refinement equation is $D^{n+1} f_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j} D^n f_n(j)$ where $f_n(i) \in \mathbb{R}^2$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ and A_i a mask of 2×2 matrices. The convergence we are interested in is C^1-convergence. In Section 6, the Hermite scheme is transformed into an associated vector scheme with $d = 2$.

In Section 7, for a given Hermite scheme we introduce the basic matrix function Φ, a 2×2 matrix. If the scheme is C^1-convergent, then $D\Phi(x/2) = \sum_{j\in\mathbb{Z}} \Phi(x - j) A_j$. Section 8 gives some spectral properties of C^1-convergent schemes so that in Section 9 we can give necessary conditions for convergence. This is completed by a theorem which uses the results of Section 4 to give sufficient conditions to obtain the convergence. A few examples are proposed in Section 10.

2 Vector subdivision schemes

A *vector subdivision scheme* $\mathcal{S}$ *of dimension* d is defined by an initial vector function $g_0 : \mathbb{Z} \to \mathbb{R}^d$ and by a set of real $d \times d$ matrix coefficients $\{B_i : i \in \mathbb{Z}\}$, with a finite number of non-zero B_i's, generating a sequence of *refinements* $g_n : \mathbb{Z} \to \mathbb{R}^d$, $n = 1, 2, 3, \dots$, recursively by

$$g_{n+1}(i) = \sum_{j\in\mathbb{Z}} B_{i-2j} g_n(j),\ i \in \mathbb{Z}. \tag{1}$$

The control vector $g_n(i)$ is attached to the dyadic point $i/2^n$.

The set of matrices $\{B_i : i \in \mathbb{Z}\}$ is called the *mask* of the subdivision scheme $\mathcal{S}$. The *support* of $\mathcal{S}$ is the smallest interval $[\tau, \tau']$ containing $\{i : B_i \neq 0\}$. The *width* of $\mathcal{S}$ is the length $\tau' - \tau$ of the support.

We now introduce a supermatrix, a matrix whose entries are $d \times d$ matrices:

$S_{ij} = B_{i-2j}$. The matrix is doubly infinite.

$$S = \begin{pmatrix} \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots \\ \cdots & B_2 & B_0 & B_{-2} & B_{-4} & B_{-6} & \cdots \\ \cdots & B_3 & B_1 & B_{-1} & B_{-3} & B_{-5} & \cdots \\ \cdots & B_4 & B_2 & B_0 & B_{-2} & B_{-4} & \cdots \\ \cdots & B_5 & B_3 & B_1 & B_{-1} & B_{-3} & \cdots \\ \cdots & B_6 & B_4 & B_2 & B_0 & B_{-2} & \cdots \\ \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix} \tag{2}$$

On each row, there is only a finite number of nonzero matrices. The supermatrix S is the matrix representation of the refinement operator, if $g_n, n = 0, 1, 2, ...$ is a sequence of refinements, if V_n is the column vector $(g_n(i))_{i\in\mathbb{Z}}$, then $V_{n+1} = SV_n$.

As indicated by Cohen, Dyn and Levin [2], in vector subdivision schemes theory it is convenient to consider $d \times d$ matrix valued functions, generated by applying the scheme to sets of matrix control points.

Definition 1 *The* basic matrix refinements *of a vector subdivision scheme is the recursive sequence* Ψ_n *(*$\Psi_n : \mathbb{Z} \to \mathbb{R}^{d\times d}$*) defined by:*

$$\Psi_{n+1}(i) = \sum_{j\in\mathbb{Z}} B_{i-2j}\Psi_n(j), i \in \mathbb{Z}, n \in \mathbb{N}, \tag{3}$$

with $\Psi_0(i) = \delta_{0i}I$ *where* I *is the identity matrix of order* d*.*

Remark 1 $\Psi_1(i) = B_i, i \in \mathbb{Z}$.

Proposition 1 *If* S *is the supermatrix* $(B_{i-2j})_{i\in\mathbb{Z},j\in\mathbb{Z}}$*, then the* (i, j)*-entry of the supermatrix* S^n *is* $\Psi_n(i - 2^n j)$*.*

Proof: We proceed by induction on n. Firstly, $S_{ij} = B_{i-2j} = \Psi_1(i - 2^1 j)$. Then according to the relation $S^{n+1} = S \times S^n$, the (i, j)-entry of S^{n+1} is $\sum_k B_{i-2k}\Psi_n(k - 2^n j) = \sum_\ell B_{i-2^{n+1}j-2\ell}\Psi_n(\ell)$. From Equation (3), this entry is equal to $\Psi_{n+1}(i - 2^{n+1}j)$. $\square$

Corollary 2 *For all* $n \in \mathbb{N}$ *and all* $i \in \mathbb{Z}$ *and for all* $g_0 : \mathbb{Z} \to \mathbb{R}^d$*:*

$$g_n(i) = \sum_{j\in\mathbb{Z}} \Psi_n(i - 2^n j)g_0(j).$$

Corollary 3 *For all* $n, n' \in \mathbb{N}$ *and for all* $i \in \mathbb{Z}$*,*

$$\Psi_{n+n'}(i) = \sum_{j\in\mathbb{Z}} \Psi_n(i - 2^n j)\Psi_{n'}(j). \tag{4}$$

Proposition 4 *If* $[\tau, \tau']$ *is the support of* $\mathcal{S}$*, then for every* $i \notin [(2^n - 1)\tau, (2^n - 1)\tau']$*,* $\Psi_n(i) = 0$*.*

Proof: Obviously the lemma is true for $n = 0$. We proceed by induction and by contradiction. We assume that there exist $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ satisfying the properties
1) $(\forall j \notin [(2^n - 1)\tau, (2^n - 1)\tau'])$, $\Psi_n(j) = 0$,
2) $(\exists i \notin [(2^{n+1} - 1)\tau, (2^{n+1} - 1)\tau'])$, $\Psi_{n+1}(i) \neq 0$.
According to Equation (3) there exists $j \in \mathbb{Z}$ such that $B_{i-2j} \neq 0$ and $\Psi_n(j) \neq 0$. So that $\tau \leq i - 2j \leq \tau'$ and $(2^n - 1)\tau \leq j \leq (2^n - 1)\tau'$. It follows that
$(2n + 1 - 1)\tau \leq i \leq (2^{n+1} - 1)\tau'$. This is a contradiction with property 2). □

3 C^0 vector subdivision schemes

Definition 2 *We say that a vector subdivision scheme of dimension* d *is* C^0*, if for every sequence of refinements* $g_n : \mathbb{Z} \to \mathbb{R}^d$*, the sequence of piecewise linear vector valued functions generated by the vectors of polygonal lines* $\{(i/2^n, g_n(i)) : i \in \mathbb{Z}\}$ *converges uniformly on any finite interval to a vector function* $g : \mathbb{R} \to \mathbb{R}^d$*.* g *is called the* limit *of the sequence of refinements* g_n*.*

Definition 3 *A* C^0 *vector subdivision scheme is* nondegenerate *if there is at least one sequence of refinements* g_n *whose limit is* $\not\equiv 0$*.*

Lemma 5 *By the definition of nondegeneracy, there exists a sequence of refinements* g_n *such that the sequence of vectors* $g_n(0)$ *converges to a nonzero vector as* $n \to \infty$*.*

Proof: Thanks to the definition of C^0 convergence, we have come to know that there exists a sequence of refinements g_n with a limit $g \not\equiv 0$. There exists an interval $[a, b], a < b$ and a positive number ϵ such that $(\forall x \in [a, b])\ ||g(x)|| > \epsilon$. There exists an integer N such that $||g_n(i) - g(i/2^n)|| < \epsilon/2$ for every $i \in [a2^n, b2^n]$ and for every $n > N$. We choose an integer $m > N$ and an integer $i' \in \mathbb{Z}$ such that $i'/2^m \in [a, b]$. We choose as initial data $\tilde{g}_0(i) = g_m(i' + i)$. If $\tilde{g}_n$ is the sequence of refinements generated by $\tilde{g}_0$, a simple verification shows that $\tilde{g}_n(i) = g_{m+n}(i + 2^n i')$. As $n \to \infty$, the sequence $\tilde{g}_n(0)$ converge to $g(i'/2^m)$, a vector whose norm is at least equal to $\epsilon/2$.
□

Definition 4 *The* truncation *of the supermatrix* S *is the matrix*

$$[S] = (B_{i-2j})_{-\tau' \leq i \leq -\tau, -\tau' \leq j \leq -\tau}$$

where $\tau = \min\{i : B_i \neq 0\}$*,* $\tau' = \max\{i : B_i \neq 0\}$*. The* truncation *of a column vector* $V = (v(i))_{i \in \mathbb{Z}}$*, is* $[V] = (v(i))_{-\tau' \leq i \leq -\tau}$*.*

Proposition 6 *If* V *is a column vector, then* $[SV] = [S][V]$*.*

Proof: If $V = (v(i))_{i\in\mathbb{Z}}$ is a column vector with $v(i) \in \mathbb{R}^d$ we define $W = SV$ i.e $w(i) = \sum_{j\in\mathbb{Z}} S_{ij} v(j), i \in \mathbb{Z}$.
Let i be in $[-\tau', -\tau]$. For $j \notin [-\tau', -\tau]$, we have $i - 2j \notin [\tau, \tau']$ and $S_{ij} = B_{i-2j} = 0$. So that $w(i) = \sum_{j=-\tau'}^{-\tau} S_{ij} v(j)$. And we conclude that $[W] = [S][V]$. □

Corollary 7 *For any $n \in \mathbb{N}$, $[S^n] = [S]^n$.*

Let us recall a result on the powers of a matrix. This theorem can be proved using Jordan normal form.

Theorem 8 (Oldenberger [17]) *Let A be a square matrix and let $P(\lambda)$ be its characteristic polynomial. The sequence A^n converges if and only if*
1) If $P(\lambda) = 0$, $|\lambda| < 1$ or $\lambda = 1$.
2) If $P(1) = 0$, the dimension of the eigenspace $\{x : Ax = x\}$ is equal to the multiplicity of the root 1 in the equation $P(\lambda) = 0$.

Theorem 9 *In a nondegenerate C^0 vector subdivision scheme of support $[\tau, \tau']$, 1 is an eigenvalue of $[S]$ and any other eigenvalue is in the open disk $\{z : |z| < 1\}$. Any eigenvector $x = (x(i))_{i\in[-\tau',-\tau]}$ of $[S]$ with eigenvalue 1 is such that $x(i) = x(-\tau)$. The dimension of $\{x : [S]x = x\}$ is equal to the multiplicity of the root 1 of the characteristic polynomial of $[S]$.*

Proof: Starting with g_0, let $V_n = (g_n(i))_{i\in\mathbb{Z}}$. We recall that $V_{n+1} = SV_n, n \in \mathbb{N}$ so that, according to Lemma 6, $[V_{n+1}] = [S][V_n]$. If ϕ_n is the sequence of piecewise linear functions generated by the polygonal lines $\{(i/2^n, g_n(i)) : i \in \mathbb{Z}\}$, by C^0 convergence, ϕ_n converges to a continuous function g. Since $\phi_n(i/2^n) = g_n(i)$, then $\lim_{n\to\infty} g_n(i) = g(0)$ for every $i \in \mathbb{Z}$. We deduce that $[V_n]$ and $[V_{n+1}]$ converge to $v = g(0)(1, 1, \cdots, 1)^T$. We conclude that $v = [S]v$ and choosing g_0 such that $g(0) \neq 0$, we have proved that 1 is an eigenvalue of $[S]$.

The equality $[V_{n+1}] = [S][V_n], n \in \mathbb{N}$ implies that $[V_n] = [S]^n[V_0], n \in \mathbb{N}$. Since the sequence V_n converges for any V_0, the sequence $[S]^n[V_0]$ converges for any $[V_0]$ so that $[S]^n$ converges. From Theorem 8, any eigenvalue of $[S]$ other than 1 is in the open disk and the dimension of $\{x : [S]x = x\}$ is equal to the multiplicity of the root 1 of the characteristic polynomial of $[S]$. □

4 Sufficient conditions for C^0 convergence

Let us now specify two notations. If g_n is a sequence of refinements of a vector subdivision scheme $\mathcal{S}$ of dimension d, then for $k = 1, ..., d$, $g_n^{(k)}$ is the k-component of g_n. If Ψ_n is the n-th basic matrix refinement of $\mathcal{S}$, then for $k, \ell = 1, ..., d$, $\Psi_n^{k\ell}$ is the $k\ell$-entry of Ψ_n.

Definition 5 *A vector subdivision scheme* $\mathcal{S}$ *is* affine *if the vector of* $\mathbb{R}^d$ *whose all components are equal to* 1 *is a right eigenvector of both matrices* $\sum_i B_{2i}$, $\sum_i B_{2i+1}$ *with the eigenvalue* 1.

Remark 2 $\mathcal{S}$ is affine iff $SV = V$ where $V = (v(i) = e)_{i\in\mathbb{Z}}$ and $e = (1, \dots, 1)^T \in \mathbb{R}^d$. This condition will happen if and only if $\sum_{j\in\mathbb{Z}} \sum_{\ell=1}^d b_{k\ell}(i-2j) = 1, k = 1, \dots, d, i \in \mathbb{Z}$, where the matrix representation of the mask of the scheme is $\{B_i = (b_{k\ell}(i)), i \in \mathbb{Z}\}$. In that case $S^n V = V$ and $\sum_{j\in\mathbb{Z}} \sum_{\ell=1}^d \psi_n^{k\ell}(i - 2^n j) = 1$, $k = 1, \dots, d$, $n \in \mathbb{N}$ where Ψ_n is the nth basic matrix refinement of $\mathcal{S}$.

Let us introduce various quantities. If $v = (v_1, v_2, \dots, v_d)^T$, $w = (w_1, w_2, \dots, w_d)^T$ are two vectors of $\mathbb{R}^d$, we set

$$\rho(v, w) = \max\{|v_k - w_{k'}| : k, k' = 1, 2, \dots, d\},$$

$$\omega(g, h) = \sup\{\rho(g(i), g(i')) : |i - i'| \leq h\}$$

where $g : \mathbb{Z} \longrightarrow \mathbb{R}^d$.

Remark 3 ω is a kind of modulus of continuity. It is obvious that for $h \geq h'$, $\omega(g, h) \geq \omega(g, h')$.

Lemma 10 *Let* $\{B_i = (b_{k\ell}(i)), k, \ell = 1, ..., d\}$ *be the mask of an affine vector subdivision scheme of dimension d, then for every sequence* g_n *of refinements, for every* $i \in \mathbb{Z}$ *and for any* $n \in \mathbb{N}$,

$$\rho(g_{n+1}(2i), g_n(i)) \leq \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{d} |b_{k\ell}(-2j)|(1 + |j|)\omega(g_n, 1).$$

and there exists $C > 0$ *such that* $\rho(g_{n+1}(2i), g_n(i)) \leq C\omega(g_n, 1)$.

Proof: In an affine vector subdivision scheme, we have $\displaystyle\sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{d} b_{k\ell}(i - 2j) = 1$. It follows that

$$g_{n+1}^{(k)}(2i) - g_n^{(k')}(i) = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{d} b_{k\ell}(-2j)(g_n^{(\ell)}(j + i) - g_n^{(k')}(i)),$$

where $g_n^{(k)}(i)$ is the k-entry of the vector $g_n(i) \in \mathbb{R}^d$.

Since $|(g_n^{(\ell)}(j + i) - g_n^{(k')}(i))| \leq |j|\omega(g_n, 1) + \omega(g_n, 0)$ and $\omega(g_n, 0) \leq \omega(g_n, 1)$, we get $\rho(g_{n+1}(2i), g_n(i)) \leq \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^d |b_{k\ell}(-2j)|(1 + |j|)\omega(g_n, 1)$. Now since the support of $\{B_{-2j}\}$ is finite, the last sum is finite and we obtain the second bound.

□

Definition 6 *The* n-th norming factor κ_n *of a subdivision scheme* $\mathcal{S}$ *of width* h *is*

$$\kappa_n = \max\{\sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{d} |\psi_n^{k\ell}(i-j2^n) - \psi_n^{k'\ell}(i'-j2^n)| : k,k'=1,\dots,d, |i-i'|\leq h\}$$

where $\psi_n^{k\ell}$ *is the* $k\ell$*-entry of the matrix function* Ψ_n*,* Ψ_n *being the sequence of basic matrix refinements of* $\mathcal{S}$*.*

Lemma 11 *Let* h *be the width of an affine vector subdivision scheme* $\mathcal{S}$ *and let* g_n *be a sequence of refinements, then for every* m *and* $n \in \mathbb{N}$

$$\omega(g_{m+n}, h) \leq \kappa_n \omega(g_m, h)/2.$$

Proof: Let $[\tau, \tau']$ be the support of $\mathcal{S}$. Define $h = \tau' - \tau$. Since $g_{m+n} = S^n g_m$, we have $g_{m+n}(i) = \sum_{j\in\mathbb{Z}} \Psi_n(i-2^n j) g_m(j), i \in \mathbb{Z}$ so that

$g_{m+n}^{(k)}(i) = \sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{d} \psi_n^{k\ell}(i-2^n j) g_m^{(\ell)}(j), i\in\mathbb{Z}, k=1,\dots,d$. Since the scheme is affine,

$\sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{d} \psi_n^{k\ell}(i-2^n j) = 1, i \in \mathbb{Z}.$

From these two last equalities we deduce that for any $c \in \mathbb{R}$,

$$g_{m+n}^{(k)}(i) - c = \sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{d} \psi_n^{k\ell}(i-2^n j)(g_m^{(\ell)}(j) - c), i\in\mathbb{Z}, k=1,\dots,d. \quad (5)$$

Let us consider two integers $i, i' \in \mathbb{Z}$ with $|i - i'| \leq h$. We may assume that $i \leq i'$ and we define the set $J = \{j : -\tau'(2^n-1) + i \leq 2^n j \leq -\tau(2^n-1) + i'\}$. By Proposition 4, $\Psi_n(i) = 0$ for $i \notin (2^n-1)[\tau,\tau']$ so that $\Psi_n(i-2^n j) = \Psi_n(i'-2^n j) = 0$ for any $j \notin J$.

With this remark, using equality (5) for (i,k) and (i',k') we get:

$g_{m+n}^{(k)}(i) - c = \sum_{j\in J}\sum_{\ell=1}^{d} \psi_n^{k\ell}(i-2^n j)(g_m^{(\ell)}(j) - c)$ and

$g_{m+n}^{(k')}(i') - c = \sum_{j\in J}\sum_{\ell=1}^{d} \psi_n^{k'\ell}(i'-2^n j)(g_m^{(\ell)}(j) - c).$

By a substraction, for any $c \in \mathbb{R}$ we obtain:

$g_{m+n}^{(k)}(i) - g_{m+n}^{(k')}(i') = \sum_{j\in J}\sum_{\ell=1}^{d} [\psi_n^{k\ell}(i-2^n j) - \psi_n^{k'\ell}(i'-2^n j)](g_m^{(\ell)}(j) - c)$, from which we deduce that:

$$\forall c \in \mathbb{R}, |g_{m+n}^{(k)}(i) - g_{m+n}^{(k')}(i')| \leq \kappa_n \max\{|g_m^{(\ell)}(j) - c|, j \in J, \ell = 1,\dots,d\}.$$

Now let $[a,b]$ be the smallest interval covering the set of the components of all the vectors $g_m(j), j \in J$ which is a finite set of real numbers. Choose $c = (a+b)/2$. We already know that for any $j \in J$ and any $\ell \in \{1,\dots,d\}$, $|g_m^{(\ell)}(j) - c| \leq (b-a)/2$. Since the diameter of J does not exceed h we deduce
$|b-a| \leq \omega(g_m, h))$ and we get the conclusion. □

Lemma 12 *If one of the norming factors of an affine vector subdivision scheme $\mathcal{S}$ is < 2, then for every sequence of refinements g_n*

$$\sum_{n=0}^{\infty} \omega(g_n, 1) < \infty.$$

Proof: Let h be the width of $\mathcal{S}$ and let $m > 0$ be an integer such that $\kappa_m < 2$. From Lemma 11, $\omega(g_{n+m}, h) \leq [\kappa_m/2]\omega(g_n, h)$ for every $n \in \mathbb{N}$.
By induction, it follows that $\omega(g_{qm+r}, h) \leq [\kappa_m/2]^q \omega(g_r, h)$ for $q \in \mathbb{N}$ and for $r = 0, 1, \dots, m-1$. Writing the euclidian division, $n = mq + r$, this is sufficient to obtain the convergence of the series $\sum_{n=0}^{\infty} \omega(g_n, h)$. Since $\omega(g_n, 1) \leq \omega(g_n, h)$, we get the conclusion. □

Theorem 13 *If one of the norming factors of an affine vector subdivision scheme $\mathcal{S}$ is < 2, then $\mathcal{S}$ is C^0. Moreover if ψ is the limit function of a sequence of refinements g_n, then all the components of ψ are the same.*

Proof: We assume that $[\tau, \tau']$ is the support of $\mathcal{S}$. We consider a sequence of refinements g_n of $\mathcal{S}$ and the corresponding sequence of piecewise linear vector functions ψ_n, $\psi_n(i/2^n) = g_n(i)$, $i \in \mathbb{Z}$. If $a < b$ are two integers of $\mathbb{Z}$, let $||\cdot||_\infty$ denote the uniform norm on $C[a,b]$. We will show that ψ_n converge to a function g in the uniform norm.

The values of ψ_n on $[a,b]$ are uniquely determined by the numbers $g_n(i), i \in [a2^n, b2^n]$. Since

$$(\forall i \in \mathbb{Z}) g_n(i) = \sum_{j \in \mathbb{Z}} S^n(i,j) g_0(j),$$

it follows from Lemma 4 that the restriction of ψ_n to $[a,b]$ is uniquely determined by the values of $g_0(j)$, $j \in [a-\tau', b+\tau]$. So there is no loss of generality by assuming that $g_0(j) = 0, j \notin [a-\tau', b+\tau]$.

Since the maximum of $||\psi_{n+1}(x) - \psi_n(x)||$ on $[a,b]$ is attained at a point on the $(n+1)$th mesh, then

$$||\psi_{n+1} - \psi_n||_\infty \leq \max\{M_n, M'_n\}, \tag{6}$$

where

$$\left\{ \begin{array}{lcl} M_n & = & \max\{||g_{n+1}(2i) - g_n(i))||_\infty : i \in \mathbb{Z}\} \\ M'_n & = & \max\{||g_{n+1}(2i+1) - (g_n(i) + g_n(i+1))/2)||_\infty : i \in \mathbb{Z}\}. \end{array} \right. \tag{7}$$

According to Lemma 10, there exists a number C such that

$$\rho(g_{n+1}(2i), g_n(i)) \leq C\omega(g_n, 1). \tag{8}$$

Since for every pair v, w of vectors of $\mathbb{R}^d$, $||v - w||_\infty \leq \rho(v, w)$, we get $M_n \leq C\omega(g_n, 1)$.

The vector $2g_{n+1}(2i+1) - g_n(i) - g_n(i+1)$ is the sum of the four vectors $g_{n+1}(2i) - g_n(i)$, $g_{n+1}(2i+2) - g_n(i+1)$, $g_{n+1}(2i+1) - g_{n+1}(2i)$ and $g_{n+1}(2i+1) - g_{n+1}(2i+2)$. It follows that $M'_n \leq 2M_n + 2\omega(g_{n+1}, 1)$.

After using last two inequalities and Lemma 12, we get

$$\sum_{n=0}^{\infty} ||\psi_{n+1} - \psi_n||_\infty < \infty.$$

From Weierstrass criterion, the sequence ψ_n converges uniformly to a continuous function ψ on $[a, b]$. The interval $[a, b]$ may be arbitrarily large, the scheme $\mathcal{S}$ is C^0.

The vector function ψ has d components $\psi^{(k)}, k = 1, ..., d$. Let us show that all of the components of ψ are the same. For any $\epsilon > 0$, there exists an integer N such that for every $n > N$, $\omega(g_n, 1) < \epsilon$ and $(\forall i \in \mathbb{Z})\ ||g_n(i) - \psi(i/2^n)|| < \epsilon$. If k, ℓ are between 1 and d, if $n > N$, then for every $i \in \mathbb{Z}$ $|\psi^{(k)}(i/2^n) - \psi^{(\ell)}(i/2^n)| < 3C\epsilon$. This follows from (8) and the definition of ρ. From this inequality and the fact that $\psi^{(k)}$ and $\psi^{(\ell)}$ are continuous, we infer that all the components of ψ are the same. □

We conclude this section with a kind of converse.

Theorem 14 *Let $\mathcal{S}$ be a C^0 vector subdivision scheme, we denote by $\psi^{k\ell}$ the $k\ell$-entry of the basic matrix function Ψ and we assume that for $k, \ell = 1, ..., d$ $\psi^{k\ell} = \psi^{1\ell}$. Then the sequence κ_n of norming factors of $\mathcal{S}$ converges to 0 as $n \to \infty$.*

Proof: Let h be the width of $\mathcal{S}$ and let $\epsilon > 0$. By uniform continuity, there exists a $\delta > 0$ such that $|\psi^{1\ell}(x) - \psi^{1\ell}(x')| < \epsilon$ for $\ell = 1, ..., d$ if $|x - x'| < \delta$. Let Ψ_n be the basic matrix refinements of $\mathcal{S}$. Since $\mathcal{S}$ is C^0, there exists an integer N_0 such that $|\psi_n^{k\ell}(i) - \psi^{1\ell}(i/2^n)| < \epsilon$ for $i \in \mathbb{Z}$, $k, \ell \in [1, d]$ and $n > N_0$. We choose N such that $N \geq N_0$ and $h/2^N < \delta$.

Let $n > N$, $|i - i'| \leq h$, $k, \ell \in [1, d]$ and $j \in \mathbb{Z}$, then $|\psi_n^{k\ell}(i - j2^n) - \psi_n^{k\ell}(i' - j2^n)| < 3\epsilon$. By Proposition 4, the number of $j \in \mathbb{Z}$ for which $\Psi_n(i - j2^n) \neq 0$ does not exceed h. It follows that the number of $j \in \mathbb{Z}$ for which $|\psi_n^{k\ell}(i - j2^n) - \psi_n^{k\ell}(i' - j2^n)| \neq 0$ does not exceed $2h$ and $\kappa_n < 6dh\epsilon$. The sequence κ_n converges to 0 as $n \to \infty$. □

5 Hermite subdivision schemes

Hermite subdivision schemes have been studied by Merrien [16], Dyn and Levin [8]. These schemes are non-stationary, but they can be transformed into stationary schemes. A *Hermite subdivision scheme $\mathcal{H}$ of order* 1 is a recursive scheme for computing a function $\phi : \mathbb{R} \to \mathbb{R}$ and its first derivative ϕ'. The initial state of the scheme is a vector function $\{f_0(i) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$. The first component of f_0 is a control value for ϕ, the second component is a control value for ϕ'. The sequence of *refinements* $\{f_n : \mathbb{Z} \to \mathbb{R}^2, n > 0\}$ is recursively defined through a family of 2×2 matrices $\{A_i = (a_{k\ell}(i))_{k,\ell=0,1} : i \in \mathbb{Z}\}$, a finite number of them being non-zero.

$$D^{n+1} f_{n+1}(i) = \sum_{j \in \mathbb{Z}} A_{i-2j} D^n f_n(j),\ i \in \mathbb{Z}, n \in \mathbb{N}, \tag{9}$$

where D is the diagonal matrix whose diagonal elements are $1, 1/2$.

Another way of writing the previous equation is

$$f^{(0)}_{n+1}(i) = \sum_{j \in \mathbb{Z}} [a_{00}(i-2j) f^{(0)}_n(j) + a_{01}(i-2j) f^{(1)}_n(j)/2^n], \tag{10}$$

$$f^{(1)}_{n+1}(i)/2^{n+1} = \sum_{j \in \mathbb{Z}} [a_{10}(i-2j) f^{(0)}_n(j) + a_{11}(i-2j) f^{(1)}_n(j)/2^n], \tag{11}$$

for $i \in \mathbb{Z}$, where $f^{(0)}_n(i), f^{(1)}_n(i)$ are the two components of the vector $f_n(i)$.

The family of matrices $\{A_i : i \in \mathbb{Z}\}$ is called the *mask* of the Hermite subdivision scheme $\mathcal{H}$. The *support* of $\mathcal{H}$ is the smallest interval $[\sigma, \sigma']$ containing $\{i : A_i \neq 0\}$.

A Hermite subdivision scheme is *interpolatory* if $A_0 = D$ and for all $i \in \mathbb{Z}$ with $i \neq 0$, $A_{2i} = 0$.

Definition 7 *We say that a Hermite subdivision scheme is C^1-*convergent *or more simply C^1, if for every initial vector function $f_0 : \mathbb{Z} \to \mathbb{R}^2$, there is a C^1-function $\phi : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ and for any sequence of integers i_n for which $\lim_{n\to\infty} i_n/2^n = x$,*

$$\lim_{n\to\infty} f^{(0)}_n(i_n) = \phi(x), \tag{12}$$

$$\lim_{n\to\infty} 2^n \Delta f^{(0)}_n(i_n) = \lim_{n\to\infty} f^{(1)}_n(i_n) = \phi'(x) \tag{13}$$

where $\Delta f(i) = f(i+1) - f(i)$ for $i \in \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{R}$.

The function ϕ is called the limit function *associated with the refinements f_n.*

Remark 4 If in a given interpolatory Hermite subdivision scheme, for every initial vector function $f_0 : \mathbb{Z} \to \mathbb{R}^2$, there is a C^1-function $\phi : \mathbb{R} \to \mathbb{R}$ for which

$$f^{(k)}_n(i) = \phi^{(k)}(i/2^n), k = 0, 1, i \in \mathbb{Z}, n \in \mathbb{N}$$

where $\phi^{(0)} = \phi$, $\phi^{(1)} = \phi'$, then the Hermite subdivision scheme is C^1.

Proposition 15 *Let $\mathcal{H}$ be a Hermite subdivision scheme. If for every initial vector function $f_0 : \mathbb{Z} \to \mathbb{R}^2$, the sequence $f^{(0)}_n(0)$ converges and there is a continuous function $\phi_1 : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ and for any sequence of integers i_n for which $\lim_{n\to\infty} i_n/2^n = x$,*

$$\lim_{n\to\infty} 2^n \Delta f^{(0)}_n(i_n) = \lim_{n\to\infty} f^{(1)}_n(i_n) = \phi_1(x), \tag{14}$$

then $\mathcal{H}$ *is* C^1.

The proof of the proposition is left to the reader.

6 The associated vector scheme

In this section, we will define the notion of reproduction of constants for a Hermite subdivision scheme $\mathcal{H}$. When $\mathcal{H}$ reproduces constants, we are able to associate to $\mathcal{H}$ a vector subdivision scheme $\mathcal{S}$. And there is a strong relationship between the C^1 convergence in $\mathcal{H}$ and the C^0 convergence in $\mathcal{S}$. Before that let us characterize the fact that for initial data $f_0(i) = (1,0)^T$, then $f_n(i) = (1,0)^T, i \in \mathbb{Z}, n \in \mathbb{N}$.

Definition 8 *A Hermite subdivision scheme of mask* $A_i = \begin{pmatrix} a_{00}(i) & a_{01}(i) \\ a_{10}(i) & a_{11}(i) \end{pmatrix}$ reproduces constants *if*

$$\sum_{j\in\mathbb{Z}} a_{00}(2j) = \sum_{j\in\mathbb{Z}} a_{00}(2j+1) = 1, \tag{15}$$

$$\sum_{j\in\mathbb{Z}} a_{10}(2j) = \sum_{j\in\mathbb{Z}} a_{10}(2j+1) = 0. \tag{16}$$

Definition 9 *A Hermite subdivision scheme is* nondegenerate *if for any vector* y *of* $\mathbb{R}^2$ *there exists at least one initial data* f_0 *such that* $\lim_{n\to\infty} f_n(0) = y$.

Proposition 16 *If a Hermite subdivision scheme is* C^1 *and nondegenerate, then constants are reproduced.*

Proof: According to Definition 9, we may consider an initial data f_0 for which $\phi(0) = 1$ where ϕ is the limit function associated with the refinements f_n. In the vector subdivision scheme with initial function $g_0 = f_0$ and mask $\{A_i\}$, the sequence of refinements is $g_n = D^n f_n$. The limit of the refinements is $g(x) = (\phi(x), 0)^T$. From equality (9), when n tends to ∞, we get that $(1,0)^T$ is an eigenvector with the eigenvalue 1 for both matrices $\sum_j A_{i-2j}$, $i = 0,1$ and we obtain both Equations (15) and (16). □

Theorem 17 (Dyn and Levin [8]) *Let* $f_n = (f_n^{(0)}, f_n^{(1)})^T$, $n = 0,1,2,\ldots$ *be the refinements of a Hermite subdivision scheme* $\mathcal{H}$ *of mask* $A_i = \begin{pmatrix} a_{00}(i) & a_{01}(i) \\ a_{10}(i) & a_{11}(i) \end{pmatrix}$, *we assume that*
$\sum_j a_{00}(2j) = \sum_j a_{00}(2j+1) = 1$ *and* $\sum_j a_{10}(2j) = \sum_j a_{10}(2j+1) = 0$.

Then the sequence $g_n(i) = (f_n^{(1)}(i), 2^n[\Delta f_n^{(0)}(i)])^T$, $n = 0, 1, 2, \ldots$ is the sequence of refinements of a vector subdivision scheme of mask

$$B_i = \begin{pmatrix} b_{00}(i) & b_{01}(i) \\ b_{10}(i) & b_{11}(i) \end{pmatrix} = 2 \begin{pmatrix} a_{11}(i) & \sum_{k=1}^{\infty} a_{10}(i-2k) \\ \Delta a_{01}(i) & \Delta \sum_{k=1}^{\infty} a_{00}(i-2k) \end{pmatrix}$$

i.e.

$$g_{n+1}(i) = \sum_{j \in \mathbb{Z}} B_{i-2j} g_n(j). \tag{17}$$

Proof: Let $[\sigma, \sigma']$ be the support of $\mathcal{H}$. Firstly, we notice that for $i < -\sigma - 1$, $B_i = 0$. Secondly, for $i > \sigma'$, by a similar argument, we have $b_{00}(i) = b_{01}(i) = 0$. Now for $k \leq 0$ we get $a_{00}(i+1-2k) = a_{00}(i-2k) = 0$ since $i+1-2k > \sigma'$ and $i-2k > \sigma'$ so that $b_{01}(i) = 2\sum_{k \in \mathbb{Z}} a_{10}(i-2k)$ and similarly $b_{11}(i) = 2\sum_{k \in \mathbb{Z}} a_{00}(i+1-2k) - a_{00}(i-2k)$. With the hypotheses, we can conclude that $B_i = 0$ for $i \notin [\sigma - 1, \sigma' + 1]$.

For $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, equation (11) gives
$f_{n+1}^{(1)}(i) = \sum_j 2a_{10}(i-2j)2^n f_n^{(0)}(j) + \sum_j 2a_{11}(i-2j) f_n^{(1)}(j)$ and these sums are finite.
We remark that
$2a_{10}(i) = 2\sum_{k=1}^{\infty}[a_{10}(i+2-2k) - a_{10}(i-2k)] = b_{01}(i+2) - b_{01}(i)$. So that

$$\begin{aligned} \textstyle\sum_j 2a_{10}(i-2j)2^n f_n^{(0)}(j) &= \textstyle\sum_j [b_{01}(i+2-2j) - b_{01}(i-2j)]2^n f_n^{(0)}(j) \\ &= \textstyle\sum_j b_{01}(i-2j)2^n f_n^{(0)}(j+1) - \sum_j b_{01}(i-2j)2^n f_n^{(0)}(j) \\ &= \textstyle\sum_j b_{01}(i-2j)2^n \Delta f_n^{(0)}(j) \end{aligned}$$

After substituting in (11), we obtain

$$f_{n+1}^{(1)}(i) = \sum_j [b_{00}(i-2j) f_n^{(1)}(j) + b_{01}(i-2j)2^n \Delta f_n^{(0)}(j)]. \tag{18}$$

Similarly, $a_{00}(i) = \sum_{k=1}^{\infty}[a_{00}(i+2-2k) - a_{00}(i-2k)]$ from which we deduce: $2[a_{00}(i+1) - a_{00}(i)] = b_{11}(i+2) - b_{11}(i)$. Using (10), we obtain

$$\begin{aligned} f_{n+1}^{(0)}(i+1) &= \textstyle\sum_j a_{00}(i+1-2j) f_n^{(0)}(j) + \sum_j a_{01}(i+1-2j) f_n^{(1)}(j) \\ f_{n+1}^{(0)}(i) &= \textstyle\sum_j a_{00}(i-2j) f_n^{(0)}(j) + \sum_j a_{01}(i-2j) f_n^{(1)}(j) \end{aligned},$$

so that

$$\begin{aligned} 2^{n+1}\Delta f_{n+1}^{(0)}(i) &= \textstyle\sum_j 2[a_{00}(i+1-2j) - a_{00}(i-2j)]2^n f_n^{(0)}(j) \\ &\quad \textstyle+ \sum_j 2[a_{01}(i+1-2j) - a_{01}(i-2j)] f_n^{(1)}(j) \\ &= \textstyle-\sum_j b_{10}(i-2j) f_n^{(1)}(j) + \sum_j [b_{11}(i+2-2j) - b_{11}(i-2j)]2^n f_n^{(0)}(j). \end{aligned}$$

This gives

$$2^{n+1}\Delta f_{n+1}^{(0)}(i) = \sum_j [b_{10}(i-2j) f_n^{(1)}(j) + b_{11}(i-2j) 2^n \Delta f_n^{(0)}(j)]. \qquad (19)$$

Formula (17) is equivalent to Formulae (18-19). □

We will say that the vector subdivision scheme whose mask is $\{B_i\}$ in the previous theorem is *associated* with the Hermite subdivision scheme of mask $\{A_i\}$.

Corollary 18 *The vector scheme associated with a nondegenerate C^1 Hermite subdivision scheme is C^0. If ϕ is the limit function associated with refinements f_n in the Hermite scheme, $(\phi', \phi')^T$ is the limit function of the refinements g_n of the associated vector scheme with $g_0(i) = (f_0^{(1)}(i), \Delta f_0^{(0)}(i))^T$.*

This follows from Lemma 16 and Theorem 17.

7 The basic matrix function

Definition 10 *The* basic matrix refinements *of a Hermite subdivision scheme is the recursive sequence of matrix functions Φ_n ($\Phi_n : \mathbb{Z} \to \mathbb{R}^{2\times 2}$):*

$$D^{n+1}\Phi_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j} D^n \Phi_n(j), i \in \mathbb{Z}, n \in \mathbb{N}, \qquad (20)$$

with $\Phi_0(i) = \delta_{0i} I$, where I is the identity matrix of order 2.

Remark 5 There is a close link between the first matrix refinement Φ_1 and the mask of a Hermite scheme: $A_i = D\Phi_1(i)$.

From Corollary 2, we deduce that for any initial data $f_0(i)$ we have

$$f_n(i) = \sum_{j\in\mathbb{Z}} \Phi_n(i - 2^n j) f_0(j), i \in \mathbb{Z}, n \in \mathbb{N}.$$

Definition 11 *If Φ_n is the sequence of basic matrix refinements of a C^1 Hermite subdivision scheme, then there exist two functions $\phi_0, \phi_1 : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ and for any sequence of integers i_n with $\lim_{n\to\infty} i_n/2^n = x$,*

$$\lim_{n\to\infty} \Phi_n(i_n) = \begin{pmatrix} \phi_0(x) & \phi_1(x) \\ \phi_0'(x) & \phi_1'(x) \end{pmatrix}. \qquad (21)$$

We define the basic matrix function *of a C^1 Hermite subdivision scheme as the matrix $\Phi(x) = \begin{pmatrix} \phi_0(x) & \phi_1(x) \\ \phi_0'(x) & \phi_1'(x) \end{pmatrix}$.*

Theorem 19 *If Φ is the basic matrix function of a C^1 Hermite subdivision scheme, then*

$$(\forall x \in \mathbb{R})\ D\Phi(x/2) = \sum_{j\in\mathbb{Z}} \Phi(x-j)A_j. \tag{22}$$

Proof: We consider a sequence of integers i_n such that $\lim_{n\to\infty} i_n/2^n = x$. From (4) and the previous remark,

$$\Phi_{n+1}(i_n) = \sum_{j=\sigma}^{\sigma'} \Phi_n(i_n - 2^n j)A_j.$$

By taking the limit as $n \to \infty$, we get (22) □

8 Spectral properties of Hermite schemes

In this section, we assume that the Hermite subdivision scheme of mask A_i is C^1 and nondegenerate. We introduce the supermatrix $H = (A_{i-2j})$. We show that 1 and 1/2 are two eigenvalues for H.

Lemma 20 *Let $z_1, z_2, ..., z_d$ be d distinct complex numbers and $p_1, p_2, ..., p_d$ be d nonzero polynomials, we assume that the sequence $u_n = \sum_{k=1}^{d} p_k(n)z_k^n$, $n = 0, 1, 2, ...$ converges to L. If $L \neq 0$, then there is an integer $j \in [1, d]$ such that $z_j = 1$, the degree of p_j is 0 and for every other integer $k \in [1, d]$, $|z_k| < 1$. If $L = 0$, then for every integer $k \in [1, d]$, $|z_k| < 1$.*

Proof: We can suppose that $|z_1| \leq |z_2| \leq \ldots \leq |z_d|$ and if $|z_k| = |z_{k+1}|$ then $\deg(p_k) \leq \deg(p_{k+1})$. Let $\rho = |z_d|$ and $\nu = \deg(p_d)$. We suppose that $|z_j| = \ldots = |z_d|$ and $\deg(p_j) = \ldots = \deg(p_d)$ with $|z_{j-1}| < |z_j|$ or ($|z_{j-1}| = |z_j|$ and $\deg(p_{j-1}) < \deg(p_j)$). Writing c_k the coefficient of $x^{\deg(p_k)}$ in p_k, the sequence u_n is asymptotically equal to the sequence $v_n = \displaystyle\sum_{k=j}^{d} c_k n^\nu z_k^n$ which converges to L.

Let us define $w_n = v_n/(n^\nu z_d^n)$. Since $|\displaystyle\sum_{\ell=0}^{n} e^{i\ell\theta}| \leq 2/|1-e^{i\theta}|$ for $\theta \neq 2k\pi, k \in \mathbb{Z}$, the arithmetical means $(w_0 + w_1 + \ldots + w_n)/(n+1)$ converges to c_d. Neither $\rho > 1$ nor ($\rho = 1$ and $\nu > 0$) may occur. Otherwise the sequence w_n would converge to 0, this would imply that $c_d = 0$ by the Cesàro theorem. But $c_d \neq 0$ since $p_d \not\equiv 0$.

First case: $\rho < 1$, then the sequence v_n converges to 0 and $L = 0$.

Second case: $\rho = 1$ and $\nu = 0$. If v_n converges to 0 then the sequence $w_n = v_n/z_d^n$ and the arithmetical means $(w_0 + w_1 + \ldots + w_n)/(n+1)$ converge to $c_d = 0$ which is impossible. So that $L \neq 0$. Now the arithmetical means $(v_0 + v_1 + \ldots + v_n)/(n+1)$

converges to L. Again, since $|\sum_{\ell=0}^{n} e^{i\ell\theta}| \leq 2/|1-e^{i\theta}|$ for $\theta \neq 2k\pi, k \in \mathbb{Z}$, there exists $j_0 \in \{j, \ldots, d\}$ such that $z_{j_0} = 1$. This j_0 is unique because the z_k are distinct and the sequence $(v_0 + v_1 + \ldots + v_n)/(n+1)$ converges to $L = c_{j_0}$. Suppose that there exists $j_1 \in \{j, \ldots, d\}$ with $j_1 \neq j_0$. Let $x_n = (v_n - L)/z_{j_1}^n$. Then the sequence x_n converges to 0 and the arithmetical means $(x_0 + x_1 + \ldots + x_n)/(n+1)$ converges to 0 and to c_{j_1} which is impossible. To conclude this case, $L \neq 0$, there exists a unique z_k with $|z_k| = 1$ and $z_k = 1$.

□

Theorem 21 *We consider a nondegenerate C^1 Hermite subdivision scheme with mask $\{A_i\}$ and support $[\sigma, \sigma']$. If $[H] = (A_{i-2j}), i, j \in [-\sigma', -\sigma]$, then 1 and 1/2 are simple roots of the characteristic polynomial of $[H]$ and any other eigenvalue is in the disk $|\lambda| < 1/2$. Moreover there is $c \in \mathbb{R}$ such that the vector function $v : \mathbb{Z} \to \mathbb{R}^2$ where $v(i) = (i + c, 1)^T$ is the eigenvector with the eigenvalue 1/2.*

Proof: Since the support is $[\sigma, \sigma']$, by using (9), we have

$$D^{n+1} f_{n+1}(i) = \sum_{j=-\sigma'}^{-\sigma} A_{i-2j} D^n f_n(j), \ -\sigma' \leq i \leq -\sigma, n \in \mathbb{N}., \tag{23}$$

If $V_n = D^n f_n(i), -\sigma' \leq i \leq -\sigma$, then $V_{n+1} = [H]V_n$. Let $m(z)$ be the minimal polynomial of $[H]$. If $V_n = (v_n(i)), -\sigma' \leq i \leq -\sigma$, $v_n(i) = (v_n^{(0)}(i), v_n^{(1)}(i))^T$, then for every $k \in \{0, 1\}$, for every $i \in [-\sigma', -\sigma]$, the sequence $v_n^{(k)}(i)$ $n = 0, 1, 2, ...$ satisfies a finite difference equation whose characteristic equation is $m(z)$. In order to show that, we write $m(z) = z^\nu - \sum_{j=0}^{\nu-1} a_j z^j$, and we have $[H]^\nu V_n = \sum_{j=0}^{\nu-1} a_j [H]^j V_n$. We deduce that $v_{n+\nu}^{(k)}(i) = \sum_{j=0}^{\nu-1} a_j v_{n+j}^{(k)}$ for $i \in [-\sigma', -\sigma]$ and $k = 0, 1$. This gives the result.

If the roots of m are λ_ℓ, $\ell = 0, ..., \nu$, then for every $k \in \{0, 1\}$, $\ell = 0, ..., \nu$ and $i \in [-\sigma', -\sigma]$, we can find polynomials $p_{k,\ell,i}(n)$ depending on the initial data such that

$$f_n^{(k)}(i)/2^{kn} = \sum_{\ell=0}^{\nu} p_{k,\ell,i}(n) \lambda_\ell^n, \quad n = 0, 1, ... \tag{24}$$

Now, we choose a sequence of refinements f_n for which the associated limit function ϕ is such that $\phi(0) = 1, \phi'(0) = 0$. As the sequence $f_n^{(0)}(0)$ converge to 1 when $n \to \infty$, we get $\lim_{n\to\infty} \sum_{\ell=0}^{\nu} p_{0,\ell,0}(n) \lambda_\ell^n = 1$. From Lemma 20, one eigenvalue of $[H]$ is 1.

Similarly, for a sequence of refinements f_n for which the associated limit function ϕ is such that $\phi(0) = 0, \phi'(0) = 1$. For any $i \in \mathbb{Z}$, the sequence $i/2^n$ converges to 0 and $f_n^{(1)}(i)$ converge to $\phi'(0) = 1$ as $n \to \infty$. We get $\lim_{n\to\infty} \sum_{\ell=0}^{\nu} p_{0,\ell,i}(n)(2\lambda_\ell)^n = 1$. So that one eigenvalue of $[H]$ is 1/2. Notice that for the eigenvalue $\lambda_j = 1$ the polynomial $p_{1,j,i}$ is 0.

For an eigenvalue λ of $[H]$, we consider an eigenvector $V = (v(i)_{-\sigma' \leq i \leq -\sigma}$ with $v(i) = (v^{(0)}(i), v^{(1)}(i))^T \in \mathbb{R}^2$. We set $f_0(i) = v(i)$ if $i \in [-\sigma', -\sigma]$ and $f_0(i) = 0$ otherwise. If f_n are the refinements of f_0, for $k = 0, 1$, we get

$$f_n^{(k)}(i)/2^{kn} = \lambda^n v^{(k)}(i), i \in [-\sigma', -\sigma], n = 0, 1, ... \tag{25}$$

Case $|\lambda| \geq 1$.
Let i be in $[-\sigma', -\sigma]$. For $k = 1$, the convergence of the Hermite subdivision scheme gives $v^{(1)}(i) = 0$. Now with $k = 0$, we obtain $\lambda = 1$ and $v^{(0)}(i) = c$. Since $v^{(1)}(i) = 0$. we have $v^{(0)}(i) = c \neq 0$. The eigenspace $\{V : [H]V = V\}$ has dimension one.

Case $1/2 \leq |\lambda| < 1$.
By convergence of the scheme we get that $\lim_{n\to\infty} 2^n \Delta f_n^{(0)}(i) = \lim_{n\to\infty} f_n^{(1)}(i) = c_1$. We show that it is impossible that $c_1 = 0$. Otherwise we get $v^{(1)}(i) = 0$ and $\Delta v^{(0)}(i) = 0$ for $i \in [-\sigma', -\sigma]$. It follows that $v^{(0)}(i) = c_0$. This means that $[H]V = V$ and it is a contradiction with $[H]V = \lambda V$, $|\lambda| < 1$. There is no loss of generality by assuming $c_1 = 1$. From the fact $v^{(1)}(i) = 1$ and $\Delta v^{(0)}(i) = 1$. It follows that $v^{(0)}(i) = c + i$. The dimension of $\{x : [H]V = 1/2V\}$ is one.

1 and 1/2 are eigenvalues of $[H]$ and all other eigenvalue λ satisfies $|\lambda| < 1/2$. The last step of the proof is to show that 1 and 1/2 are simple roots of m.

Let $\lambda \in \{1, 1/2\}$. If $\mu > 1$ is the multiplicity of this root λ, then there is a vector $v = (v(i)), -\sigma' \leq i \leq -\sigma$ such that $([H] - \lambda I)^\mu v = 0$ and $([H] - \lambda I)^{\mu-1} v \neq 0$. This is a consequence of the primary decomposition of a vector space (see Theorem 4.2 in Lang's book [15]). We set $w = ([H] - \lambda I)^{\mu-2} v$ and $w' = ([H] - \lambda I)^{\mu-1} v$. It follows that $[H]^n w = \lambda^n w + n\lambda^{n-1} w'$.

We set $f_0(i) = w(i)$ if $i \in [-\sigma', -\sigma]$ and $f_0(i) = 0$ otherwise. If f_n are the refinements of f_0, we get

$$Df_n(i) = \lambda^n w(i) + n\lambda^{n-1} w'(i), i \in [-\sigma', -\sigma], n = 0, 1, ... \tag{26}$$

From the convergence of the sequence of vector $f_n(i)$, we get that $w'(i) = 0$ for any i, which is impossible. This shows that 1 and 1/2 are simple roots of the minimal polynomial of $[H]$. Since for $\lambda = 1$ and 1/2, λ is a simple root of m and the dimension of $\{x : [H]V = \lambda V\}$ is one, then 1 and 1/2 are simple roots of the characteristic polynomial of $[H]$. □

For a Hermite subdivision scheme of mask A_i, we set

$$\alpha_{0r} = \sum_{i \equiv r \bmod 2} a_{00}(i) \quad , \quad \alpha_{1r} = \sum_{i \equiv r \bmod 2} a_{10}(i),$$
$$\beta_{0r} = \sum_{i \equiv r \bmod 2} [-a_{00}(i)i + 2a_{01}(i)] \quad , \quad \beta_{1r} = \sum_{i \equiv r \bmod 2} [-a_{10}(i)i + 2a_{11}(i)]$$

for $r = 0, 1$.

Lemma 22 *If the following conditions hold*

$$\alpha_{00} = \alpha_{01} = 1, \alpha_{10} = \alpha_{11} = 0, \beta_{00} = \beta_{01}, \beta_{10} = \beta_{11} = 1, \tag{27}$$

then the vector $X_0 = (x_0(i))_{i\in\mathbb{Z}}$ *with* $x_0(i) = (1,0)^T$ *is a right eigenvector for the matrix* $H = (A_{i-2j})_{i,j\in\mathbb{Z}}$ *with* 1 *as eigenvalue and the vector* $X_1 = (x_1(i))_{i\in\mathbb{Z}}$ *with* $x_1(i) = (i-\beta_{00}, 1)^T)$ *is a right eigenvector for the matrix* H *with* $1/2$ *as eigenvalue.*

In a nondegenerate C^1 *Hermite subdivision scheme of mask* A_i*, the conditions* (27) *are always satisfied.*

Proof: Let $Y_0 = HX_0$. Then $y_0(i) = \begin{pmatrix} \sum_j a_{00}(i-2j) \\ \sum_j a_{10}(i-2j) \end{pmatrix} = \begin{pmatrix} \alpha_{0r} \\ \alpha_{1r} \end{pmatrix}$, where $r \in \{0,1\}$ and $i \equiv r \bmod 2$. Under the hypotheses (27), we get $Y_0 = X_0$ and $HX_0 = X_0$.

If $Y_1 = HX_1$ then for any $i \in \mathbb{Z}$ and $i \equiv r \bmod 2$,

$$\begin{aligned} y_1(i) &= \begin{pmatrix} \sum_j [a_{00}(i-2j)(j-\beta_{00}) + a_{01}(i-2j)] \\ \sum_j [a_{10}(i-2j)(j-\beta_{00}) + a_{11}(i-2j)] \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sum_j [-a_{00}(i-2j)(i-2j) + 2a_{01}(i-2j)] + \sum_j a_{00}(i-2j)(i-2\beta_{00}) \\ \sum_j [-a_{10}(i-2j)(i-2j) + 2a_{11}(i-2j) + \sum_j a_{10}(i-2j)(i-2\beta_{00}) \end{pmatrix} . \\ &= \frac{1}{2}\begin{pmatrix} \beta_{0r} + (i-2\beta_{00})\alpha_{0r} \\ \beta_{1r} + (i-2\beta_{00})\alpha_{1r} \end{pmatrix} \end{aligned}$$

Under the hypotheses (27), we get $Y_1 = X_1/2$ and $HX_1 = X_1/2$.

Conversely, if the scheme is C^1 and non degenerate. Let $X_0 = (x_0(i))_{i\in\mathbb{Z}}$ with $x_0(i) = (1,0)^T$ and $[X_0] = (x_0(i))_{-\sigma'\le i\le -\sigma}$. By the previous Theorem, $[X_0]$ is a right eigenvector for the matrix $[H] = (A_{i-2j})_{-\sigma'\le i,j\le -\sigma}$ with 1 as eigenvalue. This implies that X_0 is an eigenvector of H with eigenvalue 1. $X_0 = HX_0$ gives $x_0(i) = \begin{pmatrix} \alpha_{0r} \\ \alpha_{1r} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $r \in \{0,1\}$ and $i \equiv r \bmod 2$. So that $\alpha_{00} = \alpha_{01} = 1$, $\alpha_{10} = \alpha_{11} = 0$.

Similarly $X_1 = (x_1(i))_{i\in\mathbb{Z}}$ with $x_1(i) = (i+c, 1)^T)$ is a right eigenvector for the matrix H for the eigenvalue $1/2$ and $HX_1 = 1/2X_1$ gives

$x_1(i) = \begin{pmatrix} \beta_{0r} + (i+2c)\alpha_{0r} \\ \beta_{1r} + (i+2c)\alpha_{1r} \end{pmatrix} = \begin{pmatrix} i+c \\ 1 \end{pmatrix}$ and we conclude. □

A way to summarize these facts is to use the matrix with two columns $X = (X_0 X_1)$ and to see that $HX = XD$.

Theorem 23 *If* $\Phi(x) = \begin{pmatrix} \phi_0(x) & \phi_1(x) \\ \phi_0'(x) & \phi_1'(x) \end{pmatrix}$ *is the basic matrix function of a nondegenerate* C^1 *Hermite subdivision scheme of support* $[\sigma, \sigma']$*, then*

$$\sum_{j=\sigma}^{\sigma'} \phi_0(j) = 1, \quad \sum_{j=\sigma}^{\sigma'} [(j - c_{00})\phi_0'(j) + \phi_1'(j)] = 1.$$

Proof: For $n = 1, 2, ...,$ $H^n X_0 = X_0$. From Lemma 1, we get

$$\sum_j D^n \Phi_n(i - 2^n j) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We set $i = 0$ and we let n tends to ∞. We get $\sum_{j=\sigma}^{\sigma'} \phi_0(j) = 1$ with Proposition 4.

For $n = 1, 2, ...,$ $H^n X_1 = X_1/2^n$, we also get

$$\sum_j [D^n \Phi_n(i - 2^n j) \begin{pmatrix} j - \beta_{00} \\ 1 \end{pmatrix}] = \begin{pmatrix} i - \beta_{00} \\ 1 \end{pmatrix} /2^n.$$

We set $i = 0$ and we let n tend to ∞ and we get $\sum_{j=\sigma}^{\sigma'} [(j - \beta_{00})\phi_0'(j) + \phi_1'(j)] = 1$. $\square$

We introduce an infinite row vector.

$$U = (\cdots \quad \Phi(-2) \quad \Phi(-1) \quad \Phi(0) \quad \Phi(1) \quad \Phi(2) \quad \cdots),$$

with $\Phi(x) \in \mathbb{R}^{2\times 2}$. The number of nonzero components of U is finite. If we take $x = 0$ in (22), we get $UH = DU$. If $U_0 = (1, 0)U, U_1 = (0, 1)X$, then $U_0 H = U_0$ and $U_1 H = U_1/2$. U_0, U_1 are left eigenvectors of H for the respective eigenvalues $1, 1/2$. Moreover $U_0 X_0 = U_1 X_1 = 1$.

9 Necessary and/or sufficient conditions for convergence

A first necessary condition for C^1 convergence of a Hermite subdivision scheme $\mathcal{H}$ is that the sequence of powers $[H]^n$ converge. Another necessary condition is given by Corollary 18, the associated vector subdivision scheme $\mathcal{S}$ with the mask $\{B_i\}$ is C^0. We now propose a sufficient condition for C^1 convergence of $\mathcal{H}$.

Theorem 24 *Let $\mathcal{H}$ be a Hermite subdivision scheme which reproduces constants, let $\mathcal{S}$ be the associated vector subdivision scheme, we assume that $\mathcal{S}$ is affine and that one of its norming factors is < 2, then the Hermite subdivision scheme is C^1.*

Proof: Let $A_i = \begin{pmatrix} a_{00}(i) & a_{01}(i) \\ a_{10}(i) & a_{11}(i) \end{pmatrix}$ be the mask of $\mathcal{H}$. If f_n is a sequence of refinements according to $\mathcal{H}$, the sequence $g_n = (2^n \Delta f_n^{(0)}, f_n^{(1)})^T$ is a sequence of refinements according to $\mathcal{S}$. From Theorem 13, there is a function $\phi_1 : \mathbb{R} \to \mathbb{R}$ such that $(\phi_1, \phi_1)^T$ is the limit of the refinements g_n.

We now prove the convergence of the sequence $f_n^{(0)}(0)$. By using the reproduction of constants of $\mathcal{H}$, we get

$$f_{n+1}^{(0)}(0) - f_n^{(0)}(0) = \sum_{j=-M}^{M} [a_{00}(-2j)(f_n^{(0)}(j) - f_n^{(0)}(0)) + a_{01}(-2j) f_n^{(1)}(j)/2^n]$$

where $M = \max\{|i| : A_{2i} \neq 0\}$. Since for $i = -M, ..., M$, each sequence $2^n \Delta f_n^{(0)}(i)$ and $f_n^{(1)}(i)$ is bounded, then $f_{n+1}^{(0)}(0) - f_n^{(0)}(0) = O(1/2^n)$ and $f_n^{(0)}(0)$ converge as $n \to \infty$. The hypotheses of Proposition 15 are fulfilled. The sequence of refinements f_n has a limit. The Hermite scheme is C^1. □

Remark 6 If $\mathcal{H}$ reproduces constants, then by Lemma 22, $\mathcal{S}$ is affine if only if

$$\sum_j A_{i-2j} \begin{pmatrix} i - \beta_{00} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i - \beta_{00} \\ 1 \end{pmatrix}, \quad i = 0, 1$$

where $\beta_{00} = \sum_{i \in \mathbb{Z}} [-a_{00}(2i)2i + 2a_{01}(2i)]$. If $\mathcal{S}$ is not affine, then $\mathcal{H}$ is degenerate or is not C^1, there is no other choice.

10 Examples of Hermite subdivision scheme

Example 1. We consider the one-parameter family of Hermite subdivision schemes

$$\begin{aligned} D^{n+1} f_{n+1}(2i) &= A_0 D^n f_n(i) + A_{-2} D^n f_n(i+1) \\ D^{n+1} f_{n+1}(2i+1) &= A_1 D^n f_n(i) + A_{-1} D^n f_n(i+1) \end{aligned}$$

where the nonzero matrices $A_i, -2 \leq i \leq 1$ are respectively equal to

$$\begin{pmatrix} 0 & c/2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & -1/8 + c/2 \\ 3/4 & -1/8 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/8 \\ -3/4 & -1/8 \end{pmatrix}.$$

From Theorem 17, the sequence $g_n(i) = (2^n \Delta f_n^{(0)}(i), f_n^{(1)}(i))^T$, $n = 0, 1, 2, \ldots$ is the sequence of refinements of a vector subdivision scheme of mask B_i. The nonzero matrices of the mask are $B_i, -3 \leq i \leq 1$ which are respectively equal to

$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1/4 & 0 \end{pmatrix}, \begin{pmatrix} -1/4 & 0 \\ 1/4 - c & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1/4 & 1 \end{pmatrix}, \begin{pmatrix} -1/4 & 3/2 \\ -1/4 & 1 \end{pmatrix}.$$

The parameter $\tau' = \max\{i : B_i \neq 0\}$ is equal to 1. The parameter $\tau = \min\{i : B_i \neq 0\}$ is equal to -3 if $c \neq 0$, otherwise $\tau = -2$.

Example 1a, $c = 0$. In this case, the Hermite subdivision scheme is interpolatory and converges to the Hermite cubic spline. If f_n is a sequence of refinements according to this Hermite subdivision scheme, if ϕ is the unique cubic spline with nodes on $\mathbb{Z}$ such that $(\forall n \in \mathbb{Z})(\phi(n), \phi'(n))^T = f_0(n)$, then ϕ is the limit of the refinements and the scheme is convergent. The first 5 values of $\kappa_n, n \geq 1$ are $3.5000, 3.6250, 3.0313, 2.0469, 1.1680$. Since $\kappa_5 < 2$, we get the numerical confirmation that the scheme is convergent.

The truncated matrix $[H] = (A_{i-2j}), -1 \leq i \leq 1, -1 \leq j \leq 1$ is a square matrix with 6×6 real entries. The characteristic polynomial of $[H]$ is $(\lambda - 1)(\lambda - 1/2)(\lambda -$

$1/4)^2(\lambda - 1/8)^2$. Any eigenvalue other than 1 and 1/2 is in the unit disk $|\lambda| < 1/2$. The eigenvector with the eigenvalue 1/2 is $(-1, 1, 0, 1, 1, 1)^T$.

Example 1b, $c = 1/16$**.** In this case, the Hermite subdivision scheme is not interpolatory. The first 5 values of $\kappa_n, n \geq 1$ are $4, 3.6250, 3.4766, 2.5547, 1.6875$. Since $\kappa_5 < 2$, we get the numerical confirmation that the scheme is convergent.

The truncated matrix $[H]$ is of order 8 and its characteristic polynomial is $(\lambda - 1)(\lambda - 1/2)(\lambda - 1/4)^2(\lambda - 1/8)^2\lambda^2$. The eigenvector with the eigenvalue 1/2 is $(-17/16, 1, -1/16, 1, 15/16, 1, 31/16, 1))^T$ (as predicted by Theorem 21).

Let $\Phi(x) = \begin{pmatrix} \phi_0(x) & \phi_1(x) \\ \phi_0'(x) & \phi_1'(x) \end{pmatrix}$ be the basic matrix of the Hermite subdivision scheme corresponding to the parameter $c = 1/16$, in Figure 1, we plot the graphs of the four functions ϕ_0 (left, up), ϕ_1 (right,up) , ϕ_0' (left, bottom), ϕ_1' (right, bottom).

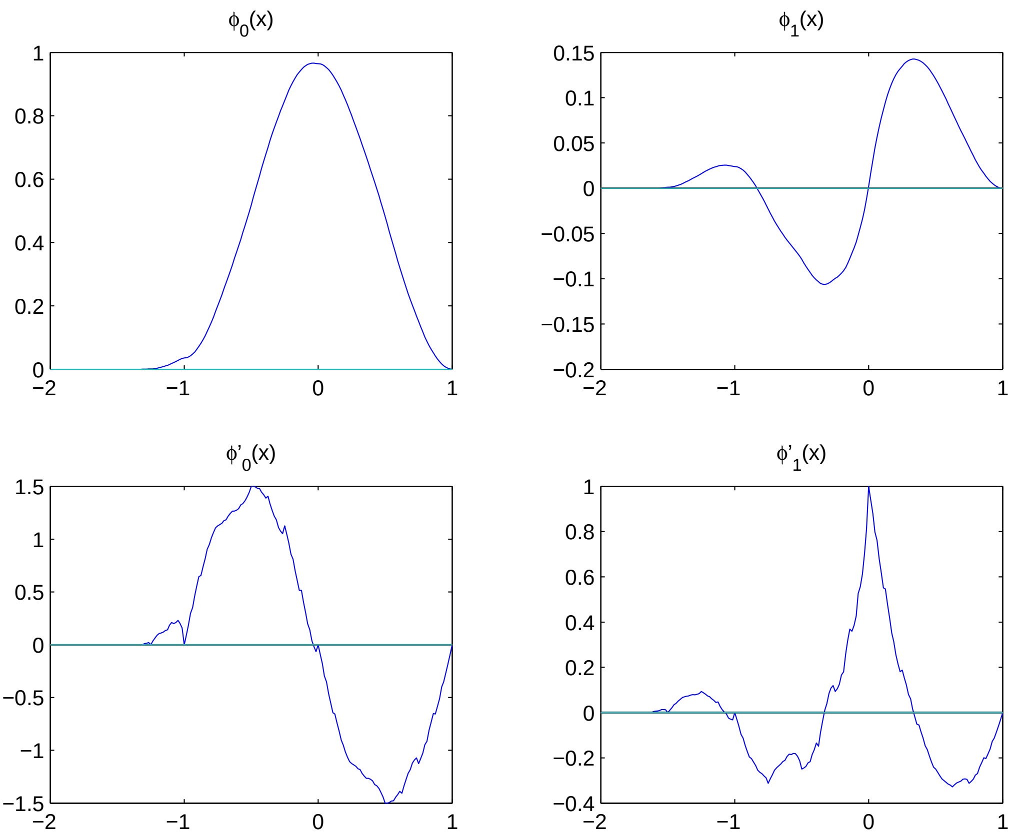

Figure 1: Basic matrix function of a non-interpolatory Hermite subdivision scheme

Example 2. We consider the one-parameter family of Hermite subdivision schemes. The matrices of the mask are $A_0 = \begin{pmatrix} 1-u & 0 \\ 0 & 1/2 - u/4 \end{pmatrix}$, $A_\epsilon = \begin{pmatrix} 243/512 & 81\epsilon/512 \\ -405\epsilon/512 & -81/512 \end{pmatrix}$,

$A_{2\epsilon} = \begin{pmatrix} u/2 & 0 \\ 0 & u/8 \end{pmatrix}$, $A_{3\epsilon} = \begin{pmatrix} 13/512 & 3\epsilon/512 \\ -5\epsilon/512 & -1/512 \end{pmatrix}$ for $\epsilon = \pm 1$. Otherwise $A_i = 0$.

Example 2a, $u = 0$. In this case, the Hermite subdivision scheme is interpolatory and one of its main properties is that any polynomial p of degree ≤ 7 is *reproduced*, i.e. the sequence of refinements of $f_0(i) = \begin{pmatrix} p(i) \\ p'(i) \end{pmatrix}$ is $f_n(i) = \begin{pmatrix} p(i/2^n) \\ p'(i/2^n) \end{pmatrix}$. We already described this example in a previous paper [6].

The first 6 norming factors are $4.5547, 4.2952, 4.2276, 3.6849, 2.3116, 1.2326$. The 6th norming factor being < 2, we get the confirmation that the scheme is C^1. The truncated matrix $[H] = (A_{i-2j}), i, j \in [-3, 3]$ is a 14×14 matrix. The eigenvalues are 1, 1/2, 1/4, 1/8 (double root), 1/16, 1/32, 1/64 (double root), 1/128 and four other unrecognized small values. The presence of the eigenvalue $1/2^k, 0 \leq k \leq 7$, comes from the fact that the function x^k is reproduced. The eigenvector with the eigenvalue 1/2 is $(-3, 1, -2, 1, -1, 1, 0, 1, 1, 1, 3, 1)^T$.

Example 2b, $u = -1/6$. In this case, the Hermite subdivision scheme is not interpolatory. The first 10 norming factors are 4.55, 5.15, 5.76, 6.17, 5.44, 5.54, 5.62. 5,84, 6,10, 6.37. No conclusion can be drawn from this sequence. Nevertheless one of the eigenvalue of the truncated matrix $[H]$ of order 14 is 0.5221, which is outside the disk $\lambda < 1/2$. From Theorem 21, this Hermite subdivision scheme cannot be C^1.

11 Comparison with another criterion of convergence

Dyn and Levin [8] found a criterion of C^1 convergence for interpolatory Hermite subdivision schemes. Let us describe this criterion. Let $\mathcal{H}$ be a Hermite subdivision scheme which reproduces constants, let $B_i = \begin{pmatrix} b_{00}(i) & b_{01}(i) \\ b_{10}(i) & b_{11}(i) \end{pmatrix}$ be the mask of its associated vector subdivision scheme $\mathcal{S}$. We assume that the associated vector subdivision scheme is affine. The subdivision matrix of $\mathcal{S}$ is $S = (s(i,j))$ where $s(2i+k, 2j+\ell) = b_{k\ell}(i-2j)$, $i \in \mathbb{Z}, j \in \mathbb{Z}$, $k, \ell = 0, 1$. By hypothesis, the matrix S is affine, $\sum_{j \in \mathbb{Z}} s(i,j) = 1$. From Proposition 10 of Daubechies et al. [5], the matrix $S' = (s'(i,j))$ defined as

$$s'_{i,j} = -\sum_{k=-\infty}^{j} (s_{i+1,k} - s_{i,k}),$$

is the subdivision matrix of a subdivision scheme $\mathcal{S}'$. Let $s'_n(i,j)$ be the ij-entry of the n-th power of S', then we define the sequence

$$\nu_n = \max\{\sum_{j \in \mathbb{Z}} |s'_n(i,j)| : i = 0, 1, 2, 3\}.$$

Then $\mathcal{H}$ is C^1 if and only if there exists an integer n for which $\nu_n < 1$ (see Theorem 3 of [9]).

For each example of the previous section, we give the first numbers ν_n. In example 1a, $\nu_1 = 1, \nu_2 = 5/8$; in example 1b, $\nu_1 = 1.1250, \nu_2 = 0.8125$; in example 2a, $\nu_1 = 1.1016, \nu_2 = 0.6776$; in example 2b, the first ten numbers are 1.43, 1.51, 1.58, 1.65, 1.72, 1.79, 1.87, 1.96, 2.043, 2.13. In these examples, the criterion of Dyn and Levin is more efficient than our criterion with the norming factors κ_n. Nevertheless, for other affine vector subdivision schemes, it may happen that the norming factors behave better. Moreover, one should point out that the criterion of convergence with the sequence κ_n can be easily extended to multivariate vector subdivision schemes while the criterion of convergence with the sequence ν_n does not have an easy extension.

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