# Monotone and convex $C^1$ Hermite interpolants generated by a subdivision scheme

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January 29, 2002

**Abstract**

We propose $C^1$ Hermite interpolants generated by the general subdivision scheme introduced by Merrien [17] and satisfying monotonicity or convexity constraints. For arbitrary values and slopes of a given function $f$ at the end-points of a bounded interval, which are compatible with the contraints, the given algorithms construct shape preserving interpolants. Moreover, these algorithms are quite simple and fast as well as adapted to CAGD. We also give error estimates in the case of interpolation of smooth functions.

*Math Subject Classification:* 41A29, 65D17.

*Keywords and Phrases:* Hermite interpolation, monotone interpolation, convex interpolation.

## 1 Introduction

The construction of monotone and/or convex smooth interpolants to monotone and/or convex data is a major research theme of approximation theory and of computer-aided geometric design. The literature on this subject is abundant and we only cite the main classes of methods, for example $C^1$ quadratic splines (Mac Allister and Roullier [14], Schumaker [21], De Vore and Lorentz [9]), $C^1$ cubic splines (Fritsch and Carlson [2]), ppf of varying degrees (Costantini [4, 5]), rational splines (Delbourgo and Gregory [8]), splines in tension (Schweikert [22], Manni [16]), parametric splines (Manni [15]), parametric spline curves (Goodman and Unsworth [13]). Some papers propose a study of more general schemes (Carnicer-Dahmen [3], Edelman-Micchelli [12]).

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On the other hand, subdivision algorithms are now hugely used in applications such as wavelets and geometric design. In particular, Hermite interpolatory subdivision schemes introduced by Merrien [17, 18] whose convergence has also been analyzed by Dyn-Levin [10, 11].

It seemed natural to study the construction of $C^1$ monotone and convex interpolants generated by these Hermite subdivision schemes. The present paper gives a general answer to this problem and it can be nicely formulated as follows: whatever be the values and slopes of an *increasing* function $f$ at the end-points of a bounded interval $[a, b]$, it is *always* possible to construct a $C^1$ *increasing* Hermite interpolant to $f$. Similarly, whatever be the values and slopes of a *convex* function $f$ at the end-points of $[a, b]$, it is *always* possible to construct a *convex* Hermite interpolant to $f$. Similar results hold for decreasing or concave functions. Of course, since the construction is local, it may be extended to an interval subdivided into subintervals with given values and slopes at the nodes of the corresponding subdivision. Therefore it is possible to construct $C^1$ shape preserving interpolants via a simple algorithm without using polynomials of high degree or other types of analytic functions.

The paper is organized as follows. In Section 2, we recall the definitions and convergence properties of Merrien's algorithm called $HS21$ . Then (Subsection 2.2 and 2.3) we introduce two interesting families of Hermite schemes defined by specific values of the parameters $(\alpha, \beta)$ which appear in the two formulae defining the algorithm. In both cases, we show that the subdivision algorithm is $C^1$-convergent and that the constructed derivative is Hölder or Lipschitz. Denoting by $VH(\alpha, \beta)$ the 4-dimensional vector space of Hermite interpolants corresponding to a given pair $(\alpha, \beta)$, we prove that $\mathbb{P}_1 \subset VH(\alpha, \beta)$ for the first family and $\mathbb{P}_2 \subset VH(\alpha, \beta)$ in the second one, where $\mathbb{P}_n$ is the vector space of polynomials of degree at most $n$.

In Section 3, we study the monotonicity of the elements of $VH(\alpha, \beta)$. For the first family, we characterize the monotonicity region and we give a simple general algorithm $(HM)$ to construct a monotone interpolant $f \in VH(\alpha, \beta)$ to arbitrary data $\{y_0, y'_0 y_1, y'_1\}$ at the boundaries of the interval $[0, 1]$ satisfying $y_0 \le y_1$ and $y'_0 \ge 0, y'_1 \ge 0$. For the second family, we only give some partial results.

In Section 4, we study the convexity of the elements of $VH(\alpha, \beta)$. For the first family, we characterize the convexity region and we give a simple general algorithm $(HC)$ to construct a convex interpolant $f \in VH(\alpha, \beta)$ to arbitrary data $\{y_0, y'_0, y_1, y'_1\}$ at the boundaries of the interval $[0, 1]$ satisfying $y'_0 \le y'_1$.

In both cases, the algorithms are illustrated by some examples.

Finally, in Section 5, we give simple error estimates for $|Hf - f|$ where $f$ is a given function of class $C^2$ or $C^3$ and $Hf$ is the Hermite interpolant of $f$ on $[0, 1]$.

In forthcoming papers, we will define Bernstein bases for the space $VH(\alpha, \beta)$ and a geometrical approach of these results. Our aim is also to extend the results to $C^2$ Hermite interpolants and to bivariate $C^1$ interpolants.

## 2 The Hermite subdivision scheme $HS21$

### 2.1 Construction and convergence

Let us recall Merrien's construction [17]. Suppose that the function $f$ and its first derivative $p$ are given at 0 and 1 and denote their values by $\{y_0, y'_0, y_1, y'_1\}$: $f$ and $p$ are built on $[0,1].2^{-n}$ by induction. At step $n$, set $h = 2^{-n}$ and
$D_n = \{x = jh, j = 0, \cdots, 2^n - 1\}$. For two consecutive points $a = jh$ and $b = (j+1)h$ of $D_n$, $f$ and $p$ are evaluated at the midpoint $m = (a+b)/2$ of $[a,b]$ by the formulae:

$$\left.\begin{array}{rcl} f(m) &=& \dfrac{f(a)+f(b)}{2} + \alpha h[p(b)-p(a)] \\ p(m) &=& (1-\beta)\dfrac{f(b)-f(a)}{h} + \beta\dfrac{p(a)+p(b)}{2} \end{array}\right\} \quad (1)$$

Hence $f$ and $p$ are defined on $D_{n+1}$. The construction depends on a pair of parameters $(\alpha, \beta)$. Reiterating the process we define $f$ and $p$ on the set of dyadic numbers $D = \bigcup D_n$ which is dense in $[0,1]$. For some values of the parameters, the functions $f$ and $p$ may be uniformly continuous on $D$ so that they may be extended to $[0,1]$. Sometimes we have in addition $p = f'$: in this case the algorithm is said to be $C^1$-*convergent*.
**Remark 1**: If $(\alpha, \beta) = (-1/8, -1/2)$, then $f$ is the Hermite interpolating cubic polynomial on $[0,1]$. If $(\alpha, \beta) = (-1/8, -1)$, then $f$ is the quadratic interpolating spline with one knot at $1/2$.

The next proposition can be proved by induction.

**Proposition 1** *The construction is linear w.r.t. the data, i.e for a given pair $(\alpha, \beta)$,if we build $(f^i, p^i)$ on $D$ from the data $\{y_0^i, {y'}_0^i, y_1^i, {y'}_1^i\}, i = 1,2$, then for any $(a,b) \in \mathbb{R}^2$, the functions built from the data $\{ay_0^1 + by_0^2, \ldots, a{y'}_1^1 + b{y'}_1^2\}$ are $f = af^1 + bf^2$ and $p = ap^1 + bp^2$.*

Now let us study the $C^1$-convergence. Once $f, p$ are built on $D$, we can define the vectors $U_n^i \in \mathbb{R}^2$ by:

$$U_n^i = \begin{pmatrix} p((i+1)2^{-n}) - p(i2^{-n}) \\ \dfrac{f((i+1)2^{-n}) - f(i2^{-n})}{2^{-n}} - \dfrac{p((i+1)2^{-n}) + p(i2^{-n})}{2} \end{pmatrix}, n \in \mathbb{N}, i = 0, \ldots, 2^n - 1.$$

and there exist two matrices $\Lambda_1$ and $\Lambda_{-1}$ of $\mathbb{R}^{2\times 2}$ such that
$U_{n+1}^{2i} = \Lambda_1 U_n^i$ and $U_{n+1}^{2i+1} = \Lambda_{-1} U_n^i$, where:

$$\Lambda_\varepsilon = \begin{pmatrix} \dfrac{1}{2} & \varepsilon(1-\beta) \\ \varepsilon\dfrac{8\alpha+1}{4} & \dfrac{1+\beta}{2} \end{pmatrix}, \varepsilon = \pm 1.$$

As in [18], the $C^1$-convergence is obtained provided that the following necessary and sufficient condition holds: the generalized spectral radius of the set $\Sigma = \{\Lambda_1, \Lambda_{-1}\}$ satisfies $\rho\{\Lambda_1, \Lambda_{-1}\} < 1$. An equivalent condition is the existence of a matrix norm $\|\cdot\|$ such that $\|\Lambda_\varepsilon\| < 1, \varepsilon = \pm 1$.
Moreover for this particular matrix norm, $\rho\{\Lambda_1, \Lambda_{-1}\} = \max_{\varepsilon=\pm 1}(\|\Lambda_\varepsilon\|)$.

Let us recall that if $\Sigma$ is a set of matrices in $\mathbb{R}^{n\times n}$ and if we denote by $\rho(M)$ the spectral radius of a matrix $M$, then the generalized spectral radius of $\Sigma$ is defined by:

$$\rho(\Sigma) = \limsup_{k\to+\infty}(\rho_k(\Sigma))^{\frac{1}{k}}, \rho_k(\Sigma) = \sup\{\rho(\prod_{i=1}^{k} M_i) : M_i \in \Sigma, 1 \leq i \leq k\}.$$

More details on the generalized spectral radius can be found in [1] and [6].

**Proposition 2** *If $\rho = \rho\{\Lambda_1, \Lambda_{-1}\} < 1$, then the function $f' = p$ is Hölder with exponent $-\log_2(\rho)$.*

**Proof:** If $\rho < 1$, then there exists a matrix norm $\|\cdot\|$ such that $\rho = \max_{\varepsilon=\pm 1}(\|\Lambda_\varepsilon\|) < 1$. Then $\|U_n^i\| < \|U_0^0\|\rho^n$ and by equivalence of this norm with $\|\cdot\|_\infty$, there exists $c_1 \in \mathbb{R}_+^*$ such that for all $n \in \mathbb{N}$ and for all $i \in \{0, \dots, 2^n - 1\}$, $|p((i+1)2^{-n}) - p(i2^{-n})| \leq c_1\rho^n$.

We now prove the existence of some constant $c_2$ such that for any $(x, y) \in [0, 1]^2$:

$$|x - y| \leq 2^{-n} \Rightarrow |p(x) - p(y)| \leq c_2\rho^n.$$

If $|x - y| \leq 2^{-n}$, there exists $z \in D_n$ with $|x - z| \leq 2^{-n}$ and $|y - z| \leq 2^{-n}$. Let us prove that $|p(x) - p(z)| \leq c_3\rho^n$. A similar equality holds with $y$ instead of $x$.
We can assume, for example, that $z < x$, so that $x = z + \sum_{i=1}^{\infty} \lambda_i 2^{-i-n}$ holds, with $\lambda_i = 0$ or $\lambda_i = 1$ ; for $x \in D$, the sum is finite. Let us define $z_0 = z$ and $z_j = z + \sum_{i=1}^{j} \lambda_i 2^{-i-n}$, for $j = 1, \dots, \infty$. Then $z_j = z_{j-1}$ or $z_j = z_{j-1} + 2^{-n-j}$. In both cases $|p(z_j) - p(z_{j-1})| \leq c_1\rho^{n+j}$, so that we have:

$$\begin{aligned} |p(x) - p(z)| &= |\sum_{j=1}^{\infty} p(z_j) - p(z_{j-1})| \leq \sum_{j=1}^{\infty} |p(z_j) - p(z_{j-1})| \\ &\leq c_1 \sum_{j=1}^{\infty} \rho^{n+j} = c_1\rho^n \frac{\rho}{1-\rho} = c_3\rho^n. \end{aligned}$$

With the same upper bound for $|p(y) - p(z)|$, there exists $c_2 = c_1\frac{2\rho}{1-\rho} \in \mathbb{R}_+^*$ such that: for all $(x, y) \in [0, 1]^2, |x - y| \leq \frac{1}{2^n} \Rightarrow |p(x) - p(y)| \leq c_2\rho^n$.

Now if $|x - y| \leq \frac{1}{2^n}$, then $n \geq -\log_2(|x - y|)$ and $|p(x) - p(y)| \leq c_2|x - y|^{-\log_2 \rho}$.
□

From now on, we will assume that $\alpha < 0$ and $\beta < 0$. For such a pair $(\alpha, \beta)$ giving rise to a $C^1$-convergent algorithm, we recall that $VH(\alpha, \beta)$ denotes the 4-dimensional vector space of all possible Hermite interpolants. The equation of the interpolant associated with the boundary data $\{y_0, y_0', y_1, y_1'\}$ can be written

$$f(t) = y_0\phi_0(t) + y_0'\psi_0(t) + y_1\phi_1(t) + y_1'\psi_1(t)$$

where $\{\phi_0, \psi_0, \phi_1, \psi_1\}$ is the *Hermite basis* formed by the interpolants associated respectively with the following sets of boundary data:

$$\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}.$$

These functions have elementary properties that can be easily proved:

**Proposition 3** *For $x \in [0, 1]$, we have:*

$$\phi_1(x) = \phi_0(1 - x), \phi_0(x) + \phi_1(x) = 1, \psi_1(x) = -\psi_0(1 - x).$$

Since the evaluation of the generalized spectral radius may be difficult, in the two next subsections, we will restrict our study to the two particular cases $\alpha = -1/8$ and $\alpha = \frac{\beta}{4(1-\beta)}$ with some conditions on $\beta$.

## 2.2 Case 1: $\alpha = \frac{\beta}{4(1-\beta)}, \beta \in [-1, 0[$

The matrices $\Lambda_\varepsilon$ can be simplified : $\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon(1-\beta) \\ \varepsilon\frac{\beta+1}{4(1-\beta)} & \frac{1+\beta}{2} \end{pmatrix}, \varepsilon = \pm 1.$

If $\beta = -1$, then $\alpha = -1/8$ and $\rho(\{\Lambda_1, \Lambda_{-1}\}) = 1/2$.

Otherwise, when evaluating the spectral radius of each of these matrices we get $\rho(\Lambda_1) = \rho(\Lambda_{-1}) = \max(1/2, |1 + \beta/2|)$, so that if $-1 < \beta < 0$, then $\rho(\{\Lambda_1, \Lambda_{-1}\}) \geq 1 + \beta/2$.

Now, for any positive real number $\theta$, we define the norm $\|\cdot\|_\theta$ in $\mathbb{R}^2$ by $\|(x, y)\|_\theta = |x| + \theta|y|$. One can prove that for any matrix $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$, the associated matrix norm defined by $\|M\|_\theta = \max_{\|X\|_\theta=1} \|MX\|_\theta$ is equal to $\|M\|_\theta = \max(|m_{11}| + \theta|m_{21}|, \frac{|m_{12}|}{\theta} + |m_{22}|)$.

When $-1 < \beta < 0$, let us choose $\theta = 2(1 - \beta)$, then $\|\Lambda_1\|_\theta = \|\Lambda_{-1}\|_\theta = 1 + \beta/2$ and $\rho(\{\Lambda_1, \Lambda_{-1}\}) \leq 1 + \beta/2$.

Finally, both results imply $\rho(\{\Lambda_1, \Lambda_{-1}\}) = 1 + \beta/2$.

**Proposition 4** *In the case when $\alpha = \frac{\beta}{4(1-\beta)}, \beta \in [-1, 0[$, then the algorithm is $C^1$-convergent and the function $p = f'$ is Hölder with exponent $-\log_2(1+\beta/2)$. Moreover, the space $VH(\alpha, \beta)$ contains the space $\mathbb{P}_1$ of affine polynomials.*

**Proof:** The generalized spectral radius being less than 1, the algorithm is convergent and $f'$ is Hölder with exponent $-\log(1+\beta/2)$ (Proposition 2). For the second assertion, it suffices to prove that the monomials 1 and $t$ are in $VH(\alpha,\beta)$; this is easily done by induction using formulae (1). □

**Remark 2**: For $0 > \alpha > \frac{\beta}{4(1-\beta)}, \beta \in [-1,0[$, the conditions given in Merrien [17] imply the $C^1$-convergence of $HS21$.

### 2.3 Case 2: $\alpha = -1/8$, $\beta \in [-2, 0]$

In this case, the matrices $\Lambda_\varepsilon$ can be simplified: $\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon(1-\beta) \\ 0 & \frac{1+\beta}{2} \end{pmatrix}, \varepsilon = \pm 1$. Berger and Wang [1] have given a property on the generalized spectral radius of triangular or block-triangular matrices:

**Lemma 5** *Assume that the matrices $M \in \Sigma$ are all block upper-triangular of the form* $M = \begin{pmatrix} M^{(1)} & & * \\ & \ddots & \\ 0 & & M^{(l)} \end{pmatrix}$ *where the $M^{(j)}$ are square matrices. Set $\Sigma^{(j)} = \{M^{(j)} : M \in \Sigma\}$, then $\rho(\Sigma) = \max(\rho(\Sigma^{(1)}), \ldots, \rho(\Sigma^{(l)}))$.*

We use their result to get $\rho(\Lambda_1, \Lambda_{-1}) = \max(\frac{1}{2}, |\frac{1+\beta}{2}|)$. For $-2 \leq \beta \leq 0$, we obtain $\rho = \rho(\Lambda_1, \Lambda_1) = 1/2$.

**Proposition 6** *For $\alpha = -1/8$ and $\beta \in [-2, 0]$, the algorithm is $C^1$-convergent and the function $p = f'$ is Lipschitz by Proposition 2. Moreover the space $VH(\alpha,\beta)$ contains the space $\mathbb{P}_2$ of quadratic polynomials.*

**Proof:** The algorithm is $C^1$-convergent since $\rho < 1$ and $f'$ is Lipschitz by Proposition 2. Now it suffices to prove that $VH(\alpha,\beta)$ contains the the functions $1, t$ and $t^2$. For example, for the latest, using formulae (1), we get by induction,:

$$f(\frac{a+b}{2}) = \frac{a^2+b^2}{2} - \frac{b-a}{8}(2b-2a) = (\frac{a+b}{2})^2, \text{ and}$$

$$p(\frac{a+b}{2}) = \frac{1-\beta}{b-a}(b^2-a^2) + \frac{\beta}{2}(2b+2a) = 2(\frac{a+b}{2}). \ \square$$

# 3 Monotone Interpolants

In this section, we will build monotone Hermite $C^1$ interpolants by subdivision. Using a classical model problem proposed by Carlson and Fritsch [2], with the data $\{y_0, y_0', y_1, y_1'\} = \{0, x, 1, y\}$ where $(x,y) \in \mathbb{R}_+^2$, we are looking for pairs $(\alpha,\beta)$ insuring the $C^1$-convergence of the algorithm $HS21$ to functions $f, p = f'$ satisfying $p \geq 0$ on $[0,1]$.

**Definition 7** *For* $(\alpha,\beta)$ *such that the algorithm* $HS21$ *is* $C^1$*-convergent to* $f$ *with* $p=f'$*, we define the monotonicity region*

$$M(\alpha,\beta)=\{(x,y)\in\mathbb{R}_+^2 : p\geq 0\},$$

*For* $\gamma>0$*, we define the triangular domain*

$$T(\gamma)=\{(x,y)\in\mathbb{R}_+^2 : x+y\leq\gamma\}$$

*For* $\delta>0$*, we define the strip*

$$B(\delta)=\{(x,y)\in\mathbb{R}_+^2 : -1/2\delta\leq x-y\leq 1/2\delta\}.$$

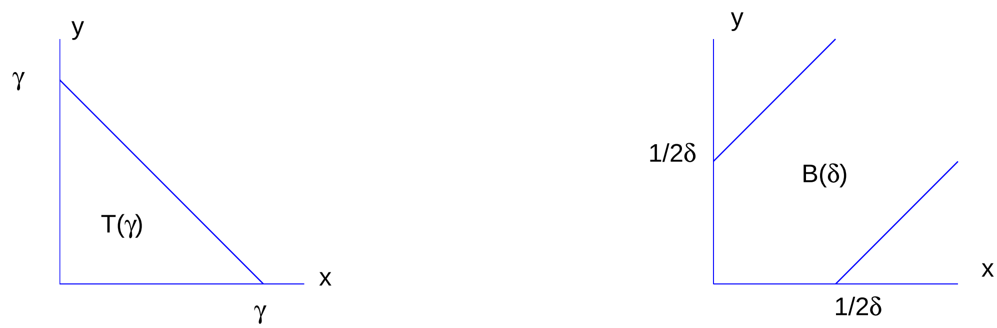


*Figure 1:* $T(\gamma)$ and $B(\delta)$

### 3.1 Properties of monotonicity regions

**Lemma 8** $M(\alpha,\beta)$ *is a convex subset of the positive cone of the plane, which is symmetrical w.r.t.* $x$ *and* $y$*, i.e.* $(x,y)\in M(\alpha,\beta)\Rightarrow (y,x)\in M(\alpha,\beta)$.

**Proof:** The convexity is a consequence of Proposition 1.
Let $f(t)=x\psi_0(t)+\phi_1(t)+y\psi_1(t)$ and $g(t)=y\psi_0(t)+\phi_1(t)+x\psi_1(t)$ be the two interpolants corresponding respectively to boundary derivatives $(x,y)$ and $(y,x)$. Since $\phi_1(t)=\phi_0(1-t)$, $\psi_0(1-t)=-\psi_1(t)$ and $\phi_0(t)+\phi_1(t)=1$, we easily obtain $g(t)+f(1-t)=1$. Therefore, $g(t)=1-f(1-t)$ and $g'(t)=f'(1-t)$, which proves that $f$ increasing implies $g$ increasing (and reciprocally). □

The next property is obvious, by definition of the sets $T(\gamma)$ and $B(\delta)$.

**Lemma 9** *If* $\gamma\leq 1/2\delta$ *then* $T(\gamma)\subset B(\delta)$.

**Remark 3**: For quadratic splines, $\alpha=-1/8,\beta=-1$, we have $M(-1/8,-1)=T(4)$ which is an immediate consequence of the condition $p(1/2)\geq 0$ (see also Edelman-Micchelli [12] and Schumaker [21]).

Now, for any pair $(\alpha,\beta)$, by formulae (1), we get $f(1/2)=1/2+\alpha(y-x)$ and $f'(1/2)=1-\beta+\frac{\beta}{2}(y+x)$. If $\alpha<0$ and $(x,y)\in M(\alpha,\beta)$, the two conditions $f(0)\leq f(1/2)\leq f(1)$ and $f'(1/2)\geq 0$ are respectively equivalent to $1/2\alpha\leq y-x\leq -1/2\alpha$ and $y+x\leq\frac{2(\beta-1)}{\beta}$. This implies, as first result, the inclusion $M(\alpha,\beta)\subset T(\frac{2(\beta-1)}{\beta})\cap B(-\alpha)$, and also the following

**Proposition 10** *If* $0 > \alpha \geq \frac{\beta}{4(1-\beta)}$ *then* $M(\alpha,\beta) \subset T(\gamma)$*, where* $\gamma = \frac{2(\beta-1)}{\beta}$.

Now, we complete this proposition:

**Theorem 11** *If* $-1 < \beta < 0$ *and* $\alpha \geq \frac{\beta}{4(1-\beta)}$ *then* $M(\alpha,\beta) = T(\gamma)$*, where* $\gamma = \frac{2(\beta-1)}{\beta}$.

**Proof:** We have to prove that $T(\gamma) \subset M(\alpha,\beta)$. Let us start by proving that $\overset{\circ}{T}(\gamma) \subset M(\alpha,\beta)$ where $\overset{\circ}{T}(\gamma)$ is the interior of $T(\gamma)$.

Let $(x,y)$ be a pair in $\overset{\circ}{T}(\gamma)$. Using the notations introduced for formulae (1), we will prove by induction that for any $n \in N$ and for any pair $(a,b)$ of consecutive points of the dyadic set $D_n$, the corresponding pairs $(f(a),f(b))$ and $(f'(a),f'(b))$ satisfy $p = (f(b)-f(a))/h > 0$, where $h = b-a = 2^{-n}$, and also $(X,Y) \in \overset{\circ}{T}(\gamma)$, where $X = f'(a)/p$ and $Y = f'(b)/p$. So that we will have $X > 0$.

The initial step is the hypothesis.

Let us suppose that the above properties are satisfied at step $n$. We have to prove the same results for the two pairs $(a,m)$ and $(m,b)$ in $D_{n+1}$ where $m = \frac{a+b}{2}$. With the above notations, we can write: $f(m) = \frac{1}{2}(f(b)+f(a)) + \alpha hp(Y-X)$ and $f'(m) = (1-\beta)p + \dfrac{\beta}{2}p(Y+X)$.

Then, we obtain
$f(m) - f(a) = \frac{1}{2}hp(1+2\alpha(Y-X)) \geq 0$ and
$f(b) - f(m) = \frac{1}{2}hp(1-2\alpha(Y-X)) \geq 0$
due to the fact that $(X,Y) \in \overset{\circ}{T}(\gamma) \subset B(-\alpha)$.
If $f(m) = f(a)$ then $Y - X = -1/2\alpha$. The only possibility would be $X = 0$ and $Y = \gamma = -1/2\alpha$. This is impossible since $\gamma \leq -1/2\alpha$ and $(X,Y) \in \overset{\circ}{T}(\gamma)$. Therefore we have $f(m) - f(a) > 0$ and similarly $f(b) - f(m) > 0$.

For the interval $[a,m]$, we set $X' = \dfrac{h}{2}\dfrac{f'(a)}{f(m)-f(a)} = \dfrac{X}{1+2\alpha(Y-X)}$ and $Y' = \dfrac{h}{2}\dfrac{f'(m)}{f(m)-f(a)} = \dfrac{(1-\beta)+\frac{\beta}{2}(Y+X)}{1+2\alpha(Y-X)}$. Then $X' > 0$ and $Y' > 0$ since $X > 0$ and $X + Y < \gamma$ which is equivalent to $(1-\beta) + \frac{\beta}{2}(Y+X) > 0$.

We must verify that

$$X' + Y' < \gamma = 2\frac{(\beta-1)}{\beta} \Leftrightarrow \frac{X + (1-\beta) + \frac{1}{2}\beta(Y+X)}{1+2\alpha(Y-X)} < 2\frac{(\beta-1)}{\beta}$$

which is equivalent to

$$A = \beta(\beta+2)X + \beta^2 Y + 8\alpha(1-\beta)(Y-X) > 2(\beta-1)(\beta+2)$$

Firstly, for $Y \geq X$, we have $8\alpha(1-\beta)(Y-X) \geq 2\beta(Y-X)$, therefore $A \geq B = \beta(\beta X + (\beta+2)Y)$. But $X > 0$ and $X + Y < \gamma$ imply

$$\beta X + (\beta+2)Y < (\beta+2)(X+Y) < \gamma(\beta+2)$$

which proves that $B > 2(\beta - 1)(\beta + 2)$ and finally that $A > 2(\beta - 1)(\beta + 2)$.

Secondly, for $Y < X$, we have $A \geq C = \beta((\beta + 2)X + \beta Y)$. But

$$(\beta + 2)X + \beta Y < (\beta + 2)(X + Y) < \gamma(\beta + 2)$$

which implies that $A \geq C > 2(\beta - 1)(\beta + 2)$, q.e.d.

On the interval $[m, b]$, the proof is similar and is not detailed here.

With this induction, we know that: if $(x, y) \in \overset{\circ}{T}(\gamma)$, then for any $a \in D$, we have $f'(a) > 0$. By construction, $f'(a)$ is a continuous function in $(x, y)$ so that if $(x, y) \in T(\gamma)$, then $f'(a) \geq 0$.

We know that the algorithm is $C^1$-convergent and we have proved that for all $n \in \mathbb{N}$ and for all $j \in \{0, \ldots, 2^n\}$, we have $f'(j2^{-n}) \geq 0$, so that $f' \geq 0$ on $[0, 1]$ and $(x, y) \in M(\alpha, \beta)$. □

**Proposition 12** *If $\beta \in [-1, 0[$, then $T(2) \in M(-1/8, \beta)$.*

**Proof:** Let $(x, y) \in T(2)$ and let us write $f$ the associated interpolant.
Then $f(0) = 0, f'(0) = x, f'(1) = y, f(1) = 1$ and
$f(1/2) = 1/2 - (y - x)/8, f'(1/2) = 1 - \beta + (x + y)/2$.

Since $T(2) \in B(1/4)$, we have:

$$\begin{array}{lcl} y \leq x + 2 & \Leftrightarrow & 4 + x - y \geq 2 \\ y \geq x - 2 & \Leftrightarrow & 4 + x - y \leq 6 \end{array}$$

For $f(1/2) = (4 - x - y)/8$, we obtain $1/4 \leq f(1/2) \leq 3/4$ which prove that
$f(0) < f(1/2) < f(1)$.

We also have $f'(1/2) > 0$. This is equivalent to: $2 - 2\beta + \beta(x + y) > 0$ or $x + y < 2(1 - 1/\beta)$ which is true since $\beta < 0$.

Now, we divide $[0, 1]$ in 2 subintervals $[0, 1/2]$ and $[1/2, 1]$.

(i) Let us define $f_1(t) = f(t/2)/f(1/2)$. It is clear that $f_1(0) = 0$ and $f_1(1) = 1$. Let $x_1 = f_1'(0)$ and $y_1 = f_1'(1)$. Then $x_1 = f'(0)/2f(1/2)$ and $y_1 = f'(1/2)/2f(1/2)$ and $(x_1, y_1) \in T(2) \Leftrightarrow f'(0) + f'(1/2) \leq 4f(1/2)$. This last inequality can be written $x + 1 - \beta + \beta(x + y)/2 \leq 2 - (y - x)/2 \Leftrightarrow (1 + \beta)(x + y) \leq 2(1 + \beta)$ which is true since $1 + \beta \geq 0$ and $0 \leq x + y \leq 2$.

(ii) Similarly, we define $f_2(t) = \dfrac{f(\frac{1+t}{2}) - \frac{1}{2}}{f(1) - f(\frac{1}{2})}$. Then $f_2(0) = 0$ and $f_2(1) = 1$. Let $x_2 = f_2'(0)$ and $y_2 = f_2'(1)$.
$(x_2, y_2) \in T(2) \Leftrightarrow 2(1 + \beta) + \beta(x + y) + 2y \leq 4 + y - x$. This last inequality can be simplify in $(1 + \beta)(x + y) \leq 2(1 + \beta)$ which is true.

By an induction, using the construction of $f$, we can prove that on each subinterval $[i2^{-n}, (i + 1)2^{-n}], i = 0, \cdots, 2^n - 1$ we keep properties similar to those of $f$ and for example $f'((2i+1)2^{-n-1}) > 0$. Thus $f$ is increasing on $D$ then on $[0, 1]$ by continuity. □

**Proposition 13** *For $\alpha = -1/8$ and $\beta \in [-1, 0[$, the functions of the Hermite basis satisfy the following properties:*
*(i) $\phi_0$ is decreasing and positive on $[0, 1]$,*
*(ii) $\psi_0$ is decreasing on $[1/2, 1]$,*
*(iii) $0 \leq \psi_0 \leq 1/4$ on $[0, 1]$,*
*and equivalent properties for $\phi_1$ and $\psi_1$.*

**Proof:**

(i) $\phi_1(0) = 0, x = \phi_1'(0) = 0, y = \phi_1'(1) = 0, \phi_1(1) = 1$. $(x, y) \in T(2)$ so that,by the preceding Proposition, $\phi_1$ is an increasing function. With $\phi_1(0) = 0$, we obtain that $\phi_1$ is positive on $[0, 1]$. Since $\phi_0(x) = \phi_1(1 - x)$, we get the first result.

(ii) We begin with $\psi_0(0) = 0, \psi_0'(0) = 1, \psi_0'(1) = 0, \psi_0(1) = 0$.
Let $f(t) = 8\psi_0(1 - t/2)$. Since $\psi_0(1/2) = 1/8$ and $\psi_0(1/2) = \beta/2$ by construction, we obtain:
$f(0) = 0, x = f'(0) = 0, y = f'(1) = -2\beta > 0, f(1) = 1$. For $\beta > -1$, we obtain $(x, y) \in T(2)$, so that $f$ is increasing on $[0, 1]$ by the preceding Proposition and $\psi_0$ is decreasing on $[1/2, 1]$. The result is completed by $\psi_0(1) = 0$. Notice that $\psi_0(1 - t)$ is positive on $[0, 1/2]$.

(iii) We already know that $\psi_0$ is positive on $[1/2, 1]$. Let us prove the positivity on $[0, 1/2]$. Let us define $g_1(t) = \psi_0(t) - \psi_0(1 - t)$.
Then $g_1(0) = g_1(1/2) = 0, g_1'(0) = 1, g_1'(1/2) = 0$. We deduce that $g_1(t) = \psi_0(2t)/2$ on $[0, 1/2]$. So that $g_1$ is positive on $[1/4, 1/2]$ by (ii). Since $\psi_0(t) = \psi_0(1 - t) + \psi_0(2t)/2$ we deduce that $\psi_0(t) \geq 0$ on $[1/4, 1/2]$. The proof of the positivity is completed by induction. Notice that $\psi_1$ is negative on $[0, 1]$.

To conclude, let us define $\omega(t) = \psi_0(t) - \psi_1(t)$. Then, using boundary conditions, we have $\omega(t) = t(1 - t)$ since $HS21$ reproduces the elements of $\mathbb{P}_2$.
Thus $0 \leq \omega(t) \leq 1/4$, so that the functions $\psi_0$ and $\psi_1$ satisfy:
$0 \leq \psi_0(t) \leq 1/4, -1/4 \leq \psi_1(t) \leq 0$. □

## 3.2 Algorithm (HM): construction of a monotone interpolant

We can assume that the interpolant is increasing,the construction being similar for a decreasing one. The boundary data are $\{y_0, y_0', y_1, y_1'\} = \{0, x, 1, y\}$ with $x \geq 0$ and $y \geq 0$.

First, we choose *a parameter* $\lambda \geq 1$ and we set $\gamma = \lambda(x + y)$.

**Case 1:** $0 \leq \gamma \leq 4$.

Since $(x, y) \in T(\gamma/\lambda) \subset T(4)$, we can interpolate by quadratic splines, i.e. we choose $(\alpha, \beta) = (-\frac{1}{8}, -1)$.

**Case 2:** $\gamma > 4$.

We choose $\alpha = -\frac{1}{2\gamma} > -\frac{1}{8}$ and $\beta = \frac{2}{2-\gamma}$, which insures that $\gamma = 2\frac{(\beta-1)}{\beta}$ and $-1 < \beta < 0$. By theorem 8, this choice implies that $(x, y) \in T(\gamma/\lambda) \subset T(\gamma) = M(\alpha, \beta)$, therefore the corresponding interpolant is increasing.

**Remark 4**: If $\lambda > 1$ we force the point $(x, y)$ to lie inside the monotonicity region. If $\lambda = 1$ and $x + y \geq 4$ then $f'(1/2) = 0$. The parameter $\lambda$ could be used as a shape parameter. In particular, in case of several data points, there could be a relation between the corresponding parameters in adjacent subintervals.

**Remark 5**: By Proposition 2, the derivative of the interpolant is Hölder with exponent $-log_2(1 + \beta/2)$.

### 3.3 Examples

In the examples below, we have choosen $\lambda = 1$ and different values of $x$ and $y$.

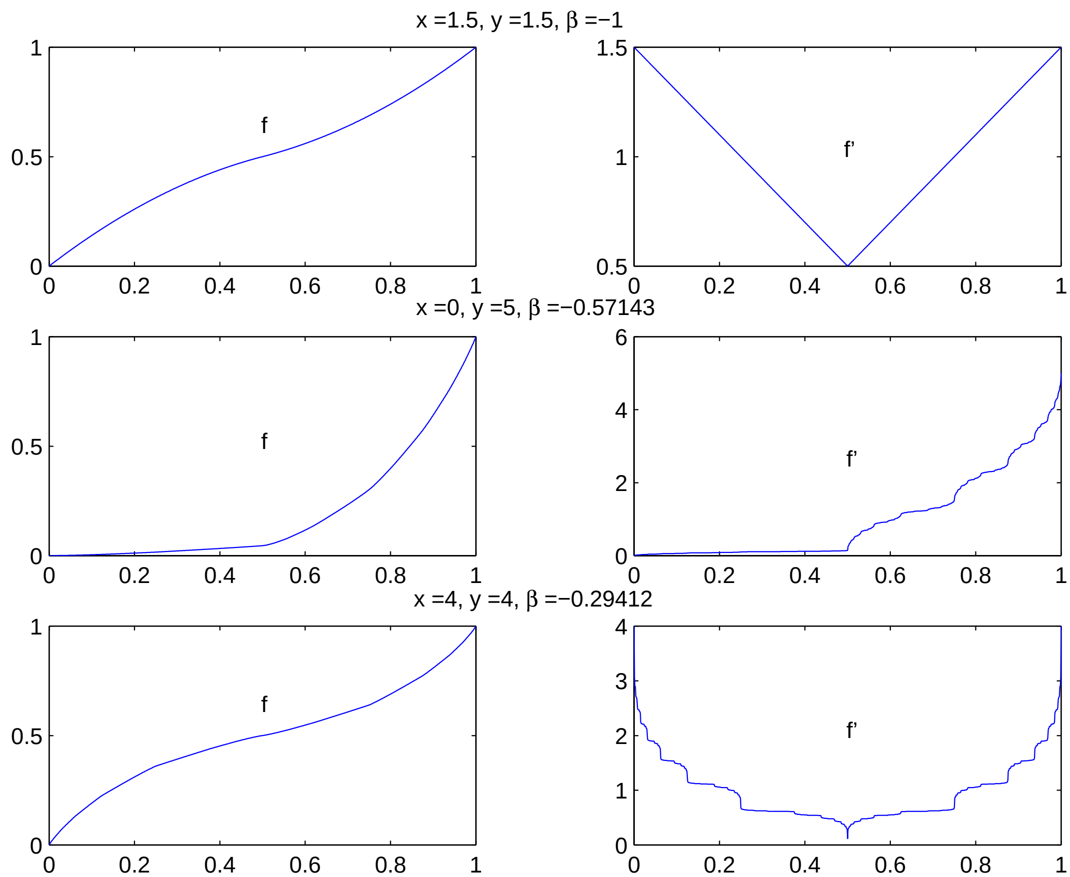


*Figure 2:* Monotone interpolants

# 4 Convex Interpolants

In this section, we build convex Hermite $C^1$ interpolants: we use the model problem associated with the boundary data $\{y_0, y_0', y_1, y_1'\} = \{0, -x, 0, y\}$, where $(x, y) \in \mathbb{R}^2$

with $x > 0$ and $y > 0$ (abbr: $(x, y) \in \mathbb{R}_+^{*2}$), and we are looking for *admissible pairs* $(\alpha, \beta)$ insuring the convergence of the subdivision algorithm to functions $f, p = f'$ with $p$ *increasing* on $[0, 1]$.

The equation of the interpolant in the Hermite basis is given by
$f(t) = -x\psi_0(t) + y\psi_1(t)$ so that if for some pair of data $(x, y)$, the interpolant $f$ is convex, then for the pair $(\lambda x, \lambda y), \lambda \geq 0$, the interpolant $\lambda f$ is also convex.

**Definition 14** *For an admissible pair $(\alpha, \beta)$, we define the convexity cone*

$$C(\alpha, \beta) = \{(x, y) \in {\mathbb{R}_+^*}^2 : p \;\; increasing\}$$

*For $\gamma \geq 1$, we define the convex cone*

$$C^*(\gamma) = \{(x, y) \in {\mathbb{R}_+^*}^2 : 1/\gamma \leq y/x \leq \gamma\}$$

**Remark 6**: For quadratic and cubic splines , we have respectively
$C(-1/8, -1) = C^*(3)$ and $C(-1/8, -1/2) = C^*(2)$, as can be verified by direct evaluation of $p'(t)$.

## 4.1 Properties of convexity cones

**Lemma 15** *For $\beta < 0$, there always holds $C(\alpha, \beta) \subset C^*(\gamma)$, where $\gamma = \frac{\beta-2}{\beta}$.*

**Proof:** Let $f$ be the interpolant associated with an admissible pair $(\alpha, \beta)$. The inequalities

$$f'(0) = -x \leq f'(1/2) = \frac{1}{2}\beta(y - x) \leq f'(1) = y$$

give respectively

$$\frac{\beta}{(\beta - 2)}x \leq y \leq \frac{(\beta - 2)}{\beta}x$$

therefore $(x, y) \in C^*(\gamma)$, with $\gamma = \frac{\beta-2}{\beta}$. $\square$

**Lemma 16** *All convexity cones have the form $C(\alpha, \beta) = C^*(\gamma)$, for some $\gamma = \gamma(\alpha, \beta)$. Equivalently, $(x, y) \in C(\alpha, \beta)$ iff $(y, x) \in C(\alpha, \beta)$.*

**Proof:** Let $f(t) = -x\psi_0(t) + y\psi_1(t)$ (resp. $g(t) = -y\psi_0(t) + x\psi_1(t)$) be the convex interpolant associated with boundary derivatives $(x, y)$ (resp. $(y, x)$). Then $g(1-t) = -y\psi_0(1-t) + x\psi_1(1-t) = y\psi_1(t) - x\psi_0(t) = f(t)$, therefore $f$ and $g$ have the same type of convexity. $\square$

**Proposition 17** *For $\beta \in [-1, 0[$, there holds $C^*(1 - 2\beta) \subset C(-1/8, \beta)$.*

**Proof:** The property is satisfied with equality for $\beta = -1$ (quadratic splines), $\beta = -1/2$ (cubic splines) in virtue of Remark 6 above. So that, we suppose that $\beta \in ]-1, 0[$.

Notice that, for all $\beta \in ]-1, 0[$, the algorithm $HS21$ converges and
$(1,1) \in C^*(1-2\beta) \cap C(-1/8, \beta)$ since the associated interpolant $f(t) = t(1-t)$ is convex.

We will prove that if $y_0$ satisfies $(1, y_0) \in C^*(1-2\beta)$, then $(1, y_0) \in C(-1/8, \beta)$. By Lemma 16, we will get $(y_0, 1) \in C(-1/8, \beta)$. Since $C^*(1-2\beta)$ and $C(-1/8, \beta)$ are cones, we have the inclusion.

*First step:* Let $(1, y_0) \in C^*(1-2\beta)$ i.e: $\dfrac{1}{1-2\beta} \le y_0 \le 1-2\beta$. Then for the interpolant $f$ with the data $\{0, -1, 0, y_0\}$, we get $f'(1/2) = \beta(y_0 - 1)/2$.

Then we have $f'(0) \le f'(1/2) \Leftrightarrow \dfrac{\beta - 2}{\beta} \ge y_0$. This last inequality is verified because $1 - 2\beta \le \dfrac{\beta-2}{\beta} \Leftrightarrow \beta^2 \le 1$ which is true.

We also have $f'(1/2) \le f'(1) \Leftrightarrow \le -\dfrac{\beta}{2-\beta} y_0$. This inequality results of
$-\dfrac{\beta}{2-\beta} \le y_0 \Leftrightarrow \beta^2 \le 1$ which is true.

Hence we have:

$$f'(0) \le f'(1/2) \le f'(1).$$

*Second step:* We now consider the restriction of $f$ to the subintervals $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$

(i) Notice that $3 - y_0 > 0$ whenever $\beta \in ]-1, 0[$ since $y_0 \le 1 - 2\beta$.

For $u \in [0,1]$, we define $g_1(u) = \dfrac{8f(u/2) + (1+y_0)u}{3-y_0}$. Then $g_1(0) = g_1(1) = 0$ since $f(1/2) = -1/8(y_0+1)$. As $g'(u) = \dfrac{4f'(u/2) + (1+y_0)}{3-y_0}$ and $f'(1/2) = \beta(y_0-1)/2$, we deduce that $g_1'(0) = -1$ and $y_1 = g_1'(1) = \dfrac{(1+2\beta)y_0 + 1 - 2\beta}{3-y_0}$. Let us prove that

$$\frac{1}{1-2\beta} \le y_1 \le 1 - 2\beta.$$

For the left inequality, $y_1 \le 1 - 2\beta \Leftrightarrow (1+2\beta)y_0 + 1 - 2\beta \le (3-y_0)(1-2\beta)$. This last inequality can be simplified in $y_0 \le 1 - 2\beta$.

For the right one, $y_1 = -(1+2\beta) + 4\dfrac{1+\beta}{3-y_0}$. Then $\dfrac{1}{1-2\beta} \le y_0$, so that it suffices to prove that $\dfrac{1}{1-2\beta} + 1 + 2\beta \le 4\dfrac{1+\beta}{3-\frac{1}{1-2\beta}} = 2\dfrac{(1+\beta)(1-2\beta)}{1-3\beta}$. This inequality can be written $2\dfrac{(1+\beta)(1-2\beta)}{1-3\beta} \ge 2\dfrac{1-2\beta^2}{1-2\beta}$; it is equivalent to $\beta^2 - \beta^3 \ge 0$ which is true since $-1 < \beta < 0$.

We have proved that the function $g_1$ can be built by subdivision and that it satisfies similar properties as $f$ i.e:
$g_1(0) = g_1(1) = 0, g_1'(0) = -1, g_1'(1) = y_1$ with $\frac{1}{1-2\beta} \leq y_1 \leq 1-2\beta$.

Notice that by Step 1:

$$g_1'(0) \leq g_1'(1/2) \leq g_1'(1) \Rightarrow f'(0) \leq f'(1/4) \leq f'(1/2).$$

(ii) From $-1 \leq \beta$, we deduce that $\dfrac{1}{1-2\beta} > \dfrac{2\beta+1}{2\beta-1}$. Since $\dfrac{2\beta+1}{2\beta-1} > y_0$ we notice that $(1-2\beta)y_0 + 2\beta + 1 > 0$.

In the same way as in (i), we define the function $g_2(u) = \dfrac{8f(\frac{1+u}{2}) + (1+y_0)(1-u)}{(1-2\beta)y_0 + 2\beta + 1}$ satisfying $g_2(0) = g_2(1) = 0$. Then, $g_2'(0) = -1$ and $g_2'(1) = y_2 = \dfrac{3y_0 - 1}{(1-2\beta)y_0 + 2\beta + 1}$.
Let us prove that

$$\frac{1}{1-2\beta} \leq y_2 \leq 1-2\beta.$$

$y_2 = \dfrac{3}{1-2\beta} - 4\dfrac{1+\beta}{(1-2\beta)y_0 + 1 + 2\beta}$ so that if $\dfrac{1}{1-2\beta} \leq y_0$, we get $\dfrac{1}{1-2\beta} \leq y_1$ which is the left inequality.

For the right one, $y_0 \leq 1-2\beta$ so that it suffices to prove that $\dfrac{3}{1-2\beta} - 4\dfrac{1+\beta}{(1-2\beta)(1-2\beta) + 1 + 2\beta} = \dfrac{3(1-2\beta) - 1}{(1-2\beta)^2 + 2\beta + 1} \leq 1-2\beta$. This last inequality can be transformed in $2 - 6\beta \leq (1-2\beta)^3 + 1 - 4\beta^2 \Leftrightarrow 0 \leq \beta^2 - \beta^3$ which is true.

The function $g_2$ satisfies similar properties as $f$ i.e:
$g_2(0) = g_2(1) = 0, g_2'(0) = -1, g_2'(1) = y_2$ with $\dfrac{1}{1-2\beta} \leq y_2 \leq 1-2\beta$ so that:

$$g_2'(0) \leq g_2'(1/2) \leq g_2'(1) \Rightarrow f'(1/2) \leq f'(3/4) \leq f'(1).$$

*Third step:* Similarly, by induction, we can built functions $g_n^i, i = 0, \cdots, 2^n - 1$ defined on $[0,1]$ by:

- $g_0^0 = f$,
- $g_1^0 = g_1, g_1^1 = g_2$,
- if $g_n^i, i = 0, \cdots, 2^n-1$ are known with $g_n^i(0) = g_n^i(1) = 0, {g_n^i}'(0) = -1, {g_n^i}'(1) = y_n^i$ with $\frac{1}{1-2\beta} \leq y_n^i \leq 1-2\beta$ , we built $g_{n+1}^{2i}(u) = \dfrac{8g_n^i(u/2) + (1+y_n^i)u}{3 - y_n^i}$ and $g_{n+1}^{2i+1}(u) = \dfrac{8g_n^i(\frac{1+u}{2}) + (1+y_n^i)(1-u)}{(1-2\beta)y_n^i + 2\beta + 1}$.
  By Step 2, the functions $g_{n+1}^{2i}$ and $g_{n+1}^{2i+1}$ keep similar properties as $f$.

So that by Step 1, we get: ${g_n^i}'(0) \le {g_n^i}'(1/2) \le {g_n^i}'(1), i = 0, \cdots, 2^n - 1$. By construction, this gives $f'(i2^{-n}) \le f'((2i+1)2^{-n-1}) \le f'((i+1)2^{-n})$. We can conclude that $f'$ is an increasing function because $f'$ is continuous. Thus $f$ is convex, which proves that $(1, y_0) \in C(-1/8, \beta)$. □

**Theorem 18** *For $\beta \in [-1, 0[$, there holds $C(\alpha, \beta) = C^*(\gamma)$, with $\gamma = \frac{\beta-2}{\beta}$ if and only if $\alpha = \frac{\beta}{4(1-\beta)}$.*

**Proof:** If $\alpha = \frac{\beta}{4(1-\beta)}$, we have to prove that $C^*(\gamma) \subset C(\alpha, \beta)$. Let $(x, y)$ be a pair in $C^*(\gamma)$. Using the notations introduced for formulae (1), we will prove by induction that for any $n \in \mathbb{N}$ and for any pair $(a, b)$ of consecutive points of the dyadic set $D_n$, the corresponding pairs $(X = -h(f'(a) - p), Y = h(f'(b) - p)$ satisfy $(X, Y) \in C^*(\gamma)$, where $p = (f(b) - f(a))/h$ and $h = b - a = 2^{-n}$.

At the initial step, $n = 0$, we have $X = x$ and $Y = y$ so that $(X, Y) \in C^*(\gamma)$.

If we suppose at step $n$, on $[a, b]$ that

$$(X, Y) \in C^*(\gamma) \Leftrightarrow \frac{\beta f'(b) + (\beta - 2)f'(a)}{2(\beta - 1)} \le \frac{f(b) - f(a)}{h} \le \frac{\beta f'(a) + (\beta - 2)f'(b)}{2(\beta - 1)},$$

we have to prove:

$$\frac{\beta f'(m) + (\beta - 2)f'(a)}{2(\beta - 1)} \le \frac{f(m) - f(a)}{h/2} \le \frac{\beta f'(a) + (\beta - 2)f'(m)}{2(\beta - 1)},$$

and

$$\frac{\beta f'(b) + (\beta - 2)f'(m)}{2(\beta - 1)} \le \frac{f(b) - f(m)}{h/2} \le \frac{\beta f'(m) + (\beta - 2)f'(b)}{2(\beta - 1)}.$$

Let us prove the first inequality:

$$\begin{aligned}
&\frac{\beta f'(m) + (\beta - 2)f'(a)}{2(\beta - 1)} \le \frac{f(m) - f(a)}{h/2} \\
&\Leftrightarrow \beta(1 - \beta)\frac{f(b) - f(a)}{h} + \frac{\beta^2}{2}(f'(b) + f'(a)) + (\beta - 2)f'(a) \\
&\qquad \ge \frac{4(\beta - 1)}{h}[\frac{f(b) - f(a)}{2} + \alpha h(f'(b) - f'(a))] \\
&\Leftrightarrow (1 - \beta)(\beta + 2)\frac{f(b) - f(a)}{h} \\
&\qquad \ge [-4\alpha(\beta - 1) - \tfrac{\beta^2}{2} + 2 - \beta]f'(a) + [4\alpha(\beta - 1) - \tfrac{\beta^2}{2}]f'(b)
\end{aligned}.$$

Now if $\alpha = \frac{\beta}{4(1-\beta)}$, the latter inequality is equivalent to

$\frac{\beta f'(b) + (\beta - 2)f'(a)}{2(\beta - 1)} \le \frac{f(b) - f(a)}{h}$ which is part of the hypotheses.

A similar method, which is not detailed here, gives the other inequalities.

With these inequalities, we have proved that at each step $n$, $(X, Y) \in C^*(\gamma)$ so that $X \ge 0, Y \ge 0$. Then $f$ is convex i.e $(x, y) \in C(\alpha, \beta)$ since for any pair $(a, b)$ of consecutive points in $D_n$, we have:

$$f'(a) \le \frac{f(b) - f(a)}{h} \le f'(b)$$

Conversely, suppose that $C(\alpha,\beta) = C^*(\gamma)$, with $\gamma = \frac{\beta-2}{\beta}$. To prove that $\alpha = \frac{\beta}{4(1-\beta)}$, we choose $x = -f'(0) = -\beta > 0$ and $y = f'(1) = 2-\beta > 0$, so that $y = \gamma x$. Then $f(1/2) = 2\alpha(1-\beta), f'(1/2) = \beta$ and $f'(1/4) = 4\alpha(1-\beta)^2 + \beta^2$. To satisfy the inequalities $f'(0) = \beta \leq f'(1/4) \leq f'(1/2) = \beta$, we must have $\beta = 4\alpha(1-\beta)^2 + \beta^2$, i.e $\alpha = \frac{\beta}{4(1-\beta)}$. □

**Proposition 19** *For $\beta \in [-1, 0[$ and $\alpha = \frac{\beta}{4(1-\beta)}$, the functions of the Hermite basis satisfy the following properties: (i) $\phi_0$, $\psi_0$ and $\psi_1$ are concave on $[0,1/2]$ and convex on $[1/2,1]$, $\phi_1$ is convex on $[0,1/2]$ and concave on $[1/2,1]$,*
*(ii) $\phi_0, \phi_1, \psi_0$ are positive on $[0,1]$, $\psi_1$ is negative.*

**Proof:** (i) It suffices to study the two functions $\phi_0$ and $\psi_0$ since the two remaining ones are defined respectively by $\phi_1(t) = \phi_0(1-t)$ and $\psi_1(t) = -\psi_0(1-t)$. The boundary values of $\phi_0$ being equal to $\{1,0,0,0\}$, we get $\phi_0(1/2) = 1/2$ and $\phi_0'(1/2) = \beta-1$. On the interval $[0,1]$, let us define $g(t) = 2-t-2\phi_0(t/2)$, so that $g$ can be built by $HS21$ from initial data: $\{0,-1,0,-\beta\}$. The concavity of $\phi_0$ on $[0,1/2]$ is equivalent to the convexity of $g$ on $[0,1]$. By theorem 18, it is satisfied iff $(1,-\beta) \in C^*(\frac{\beta-2}{\beta})$, i.e. iff

$$\frac{\beta}{\beta-2} \leq -\beta \leq \frac{\beta-2}{\beta}$$

The left inequality holds iff $\beta \leq 1$, the right one holds iff $(\beta-1)(\beta+2) \leq 0$, and both are satisfied. The convexity of $\phi_0$ on the second interval $[1/2,1]$ comes from the fact that $\phi_0(t) = 1-\phi_0(1-t)$.

The boundary values of $\psi_0$ being equal to $\{0,1,0,0\}$, we get $\psi_0(1/2) = -\alpha$ and $\psi_0'(1/2) = \frac{1}{2}\beta$. On the interval $[0,1]$, let us set $h(t) = -2\alpha t - 2\psi_0(t/2)$, so that we obtain a function $h$ which can be built with $HS21$ from initial data:
$\{0, -X = -(1+2\alpha), 0, Y = -(\frac{1}{2}\beta+2\alpha)\}$. The concavity of $\psi_0$ on $[0,1/2]$ is equivalent to the convexity of $h$ on $[0,1]$. By theorem 18, it is satisfied iff $Y/X \in C^*(\frac{\beta-2}{\beta})$, i.e. iff

$$\frac{\beta}{\beta-2} \leq \frac{-(\beta+4\alpha)}{2(1+2\alpha)} \leq \frac{\beta-2}{\beta}.$$

The left inequality is equivalent to $\beta \leq 1$ and the right one to $(\beta-1)(\beta+2)(\beta-2) \leq 0$. The proof of the convexity of $\psi_0$ on $[1/2,1]$ is similar.

(ii) With the above properties, $\phi_0'$ is decreasing on $[0,1/2]$ and increasing on $[1/2,1]$. Since $\phi_0'(0) = 0$ and $\phi_0'(1) = 0$, $\phi_0'$ is negative on $[0,1]$. So that $\phi_0$ is decreasing on $[0,1]$ and $\phi_0(1) = 0$. Then $\phi$ is a positive function.

Similarly, $\psi_0$ is an increasing function on $[0,u], 0 \leq u \leq 1/2$ and a decreasing function on $[u,1]$ with $\psi(0) = \psi(1) = 0$ so that $\psi_0 \geq 0$.

The other proofs are similar. □

## 4.2 Algorithm (HC): construction of a convex interpolant

We assume that the boundary data are $\{0, -x, 0, y\}$, with $(x, y) \in {\mathbb{R}_+^*}^2$.

- Suppose that $y/x \geq 1$. We choose a *parameter* $\lambda \geq 1$ and we set $\gamma = \lambda y/x \geq 1$.
  - **Case 1:** $1 \leq \gamma \leq 3$. Since $(x, y) \in C^*(\gamma) \subset C^*(3)$, we can interpolate by quadratic splines i.e. $\beta = -1, \alpha = -1/8$.
  - **Case 2:** $\gamma > 3$. We set $\frac{\beta-2}{\beta} = \gamma$, i.e. $\beta = \frac{-2}{\gamma-1}$ and $\alpha = \frac{\beta}{4(1-\beta)} = \frac{-1}{2(\gamma+1)}$. With this choice, we get $-1 < \beta < 0$ and $(x, y)$ lies in the cone $C^*(\gamma) = C(\alpha, \beta)$.
- Suppose that $y/x < 1$. We choose a *parameter* $\lambda \geq 1$ and we set $\gamma = \lambda x/y \geq 1$.
  - **Case 3:** $1 \leq \gamma \leq 3$. Since $(x, y) \in C^*(\gamma) \subset C^*(3)$, we can interpolate by quadratic splines.
  - **Case 4:** $\gamma > 3$. We set $\frac{\beta-2}{\beta} = \gamma$, i.e. $\beta = \frac{-2}{\gamma-1}$ and $\alpha = \frac{\beta}{4(1-\beta)} = \frac{-1}{2(\gamma+1)}$ as in Case 2.

**Remark 7**: If we choose $\lambda = 1$ in Case 2, the interpolant is linear on $[0, 1/2]$: $f(t) = -xt$; similarly in Case 4 on $[1/2, 1]$ with $f(t) = y(t-1)$. This parameter $\lambda$ can be choosen for a shape parameter.

### 4.3 Examples

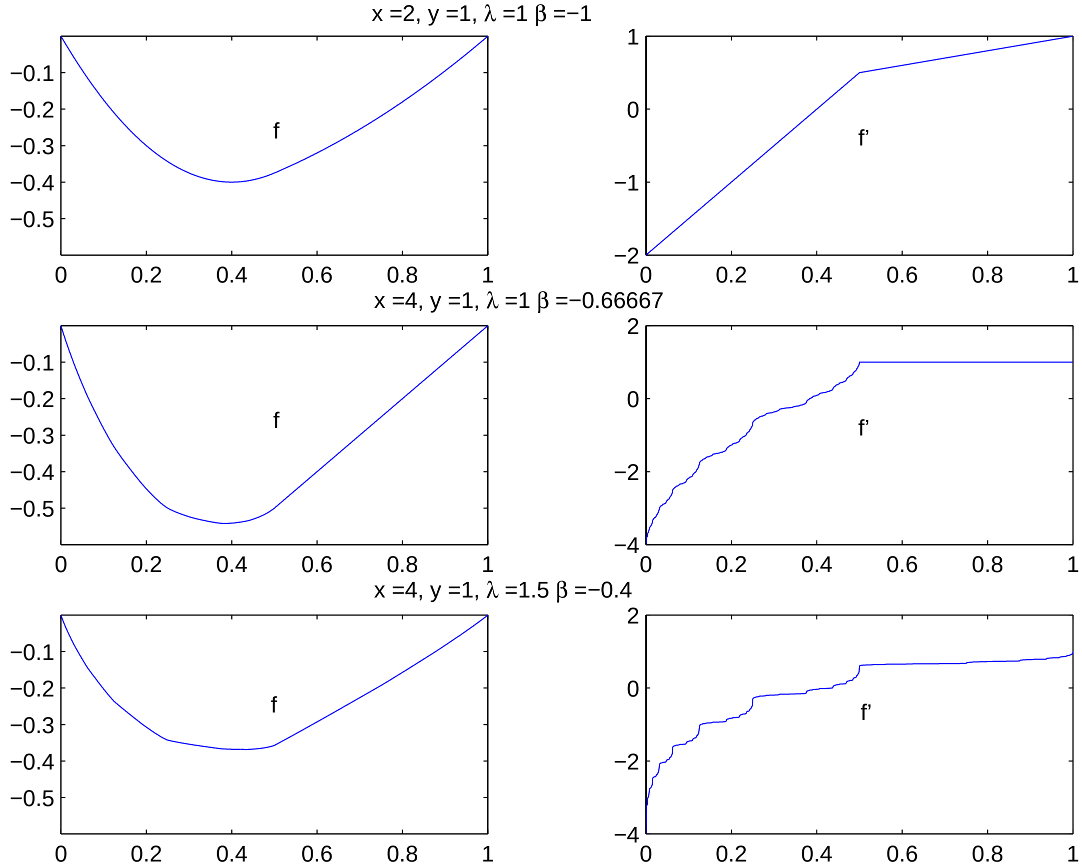


*Figure 3:* Convex interpolants

## 5 Interpolation error estimates

Let us suppose that the interpolation on $[0, 1]$ is done from the values of a given function $f$ at the end-points of the interval. We will suppose that we are in the case of $C^1$-convergence. Let us denote by $Hf$ the Hermite interpolant of $f$, i.e.

$$Hf(t) = f(0)\phi_0(t) + f'(0)\psi_0(t) + f(1)\phi_1(t) + f'(1)\psi_1(t).$$

### 5.1 Case 1: $-1 \leq \beta < 0$ and $\alpha = \frac{\beta}{4(1-\beta)}$

In this case, the vector space $VH(\alpha, \beta)$ of all Hermite interpolants associated with a given pair $(\alpha, \beta)$ contains the space of affine polynomials.

**Proposition 20** *For any function $f \in C^2[0,1]$, the following error bound holds:*

$$|f(x) - Hf(x)| \leq (1/8 + |\alpha|)\|f''\|_\infty \leq 1/4\|f''\|_\infty$$

**Proof:** Let $Af$ be the linear interpolant of $f$ defined by:

$$Af(x) = (1-x)f(0) + xf(1).$$

To get the result, we use an intermediate operator: $Hf - f = Hf - Af + Af - f$.

Firstly, if $f$ is a $C^2$ function on $[0,1]$, then, following Davis [7] the error bound holds: $\|Af - f\|_\infty \leq \frac{1}{8}\|f''\|_\infty$.

Secondly, the subdivision interpolants reproduce $\mathbb{P}_1$ thus
$x = \phi_1(x) + \psi_0(x) + \psi_1(x), 1 - x = \phi_0(x) - \psi_0(x) - \psi_1(x)$,
and $Hf(x) = f(0)\phi_0(x) + f'(0)\psi_0(x) + f(1)\phi_1(x) + f'(1)\psi_1(x)$, whence

$$\begin{aligned} Hf(x) - Af(x) &= f(0)[\phi_0(x) - (1-x)] + f'(0)\psi_0(x) \\ &\quad + f(1)[\psi_1(x) - x] + f'(1)\psi_1(x) \\ &= f(0)[\psi_0(x) + \psi_1(x)] + f'(0)\psi_0(x) \\ &\quad + f(1)[-\psi_0(x) - \psi_1(x)] + f'(1)\psi(x) \end{aligned}$$

Using both results above with $f'(0) = f'(x) - \int_0^x f''(t)dt$
and $f'(1) = f'(x) + \int_x^1 f''(t)dt$, we obtain:
$Hf(x) - Af(x) = \int_0^1 [(t-1)\psi_0(x) - t\psi_1(x)]f''(t)dt$.

We have proved in Proposition 19 that $\psi_0$ is a positive function, and we know that $\psi_1(x) = -\psi_0(1-x)$, therefore: $|Hf(x) - Af(x)| \leq \frac{1}{2}\|f''\|_\infty[\psi_0(x) - \psi_1(x)]$.

Let $\omega(x) = -\psi_0(x) + \psi_1(x)$; then $\omega(0) = \omega(1) = 0$ and $\omega'(0) = -1, \omega'(1) = 1$. Since $(1,1) \in C(\alpha,\beta)$, $\omega$ is a convex function with $\omega(x) = \omega(1-x)$, so that the minimum of $\omega$ is $\omega(1/2) = -2\psi_0(1/2) = 2\alpha$. We obtain: $|Hf(x) - Af(x)| \leq |\alpha|\|f''\|_\infty$. The last inequality follows from $|\alpha| \leq \frac{1}{8}$ for $-1 \leq \beta < 0$ and $\alpha = \frac{\beta}{4(1-\beta)}$. $\square$

## 5.2 Case 2: $-1 \leq \beta < 0$ and $\alpha = -\frac{1}{8}$

In this case, the vector space $VH(\alpha,\beta)$ of all Hermite interpolants associated with a given pair $(\alpha,\beta)$ contains the space $\mathbb{P}_2$ of quadratic polynomials.

**Proposition 21** *For any function $f \in C^3[0,1]$, the following error bound holds:*

$$|f(x) - Hf(x)| \leq \frac{2+\beta}{96}\|f^{(3)}\|_\infty \leq \frac{1}{48}\|f^{(3)}\|_\infty$$

**Proof:** Let $Qf$ be the quadratic spline interpolant of $f$, and $Hf$ the Hermite interpolant built by subdivision. We get the result by using $Q$ as intermediate operator: $Hf - f = Hf - Qf + Qf - f$.

Firstly, it is known that $\|Qf - f\|_\infty \leq \frac{1}{96}\|f^{(3)}\|_\infty$. This result can be found e.g. in Sablonnière [20] (Chapter 3).

Secondly, we bound $|Hf - Qf|$ on $[0, 1/2]$. A similar result can be obtained on $[1/2, 1]$.

We have $Hf(0) = Qf(0) = f(0)$ and $(Hf)'(0) = (Qf)'(0) = f'(0)$. Using formulae (1), we get $Hf(1/2) = Qf(1/2)$ and:
$(Hf)'(1/2) = (1-\beta)(f(1) - f(0)) + \frac{\beta}{2}(f'(1) + f'(0),$
$(Qf)'(1/2) = 2(f(1) - f(0)) - (f'(1) + f'(0),$
since the quadratic spline is obtained by $HS21$ for $\alpha = -1/8, \beta = -1$.

We define $H_1(x) = Hf(x/2)$ and $Q_1(x) = Qf(x/2)$ on $[0, 1]$. $Q_1$ is a polynomial of degree 2 and $HS21$ reproduces this polynomial. So that both functions can be expressed in the Hermite basis:

$$\begin{array}{rcl} H_1(x) & = & H_1(0)\phi_0 + H_1'(0)\psi_0 + H_1(1)\phi_1 + H_1'(1)\psi_1 \\ Q_1(x) & = & Q_1(0)\phi_0 + Q_1'(0)\psi_0 + Q_1(1)\phi_1 + Q_1'(1)\psi_1 \end{array}$$

Hence $|H_1(x) - Q_1(x)| \leq |H_1'(1) - Q_1'(1)|.\|\psi_1\|_\infty$, so that
$|Hf - Qf\|_{[0,1/2],\infty} \leq \frac{1}{2}\|(Hf)'(1/2) - (Qf)'(1/2)|.\|\psi_1\|_{[0,1],\infty}$.
Now using a Taylor expansion of $f$ and $f'$ at $1/2$, we get:
$|H_1'(1/2) - Q_1'(1/2)| = |1+\beta|.|f(1) - f(0) - \frac{1}{2}[f'(1) - f'(0)]| \leq \frac{|1+\beta|}{12}.\|f^{(3)}\|_\infty$.

We have proved in Proposition 13 that $\|\psi_1\|_\infty \leq \frac{1}{4}$ therefore
$\|Hf - Qf\|_{\infty,[0,1/2]} \leq \frac{|1+\beta|}{48}\|f^{(3)}\|_\infty$.

Finally, $1 \geq 1 + \beta \geq 0$ implies the last inequality. □

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