Rational Splines for Hermite Interpolation with Shape Constraints.

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Abstract

This paper is concerned in shape-preserving Hermite interpolation of a given function f at the endpoints of an interval using rational functions. After a brief presentation of the general Hermite problem, we investigate two cases. In the first one, f and f' are given and it is proved that for any monotonic set of data, it is always possible to construct a monotonic rational function of type [3/2] interpolating those data. Positive and convex interpolants can be computed by a similar method. In the second case, results are proved using rational function of type [5/4] for interpolating the data coming from f, f' and f'' with the goal of constructing positive, monotonic or convex interpolants. Error estimates are given and numerical examples illustrate the algorithms.

keyword: Hermite Interpolation, Rational Functions, Shape-preservation.

1 Introduction

This paper is concerned in shape-preserving Hermite interpolation of a given function f and its derivatives at the endpoints of an interval ($[0, 1]$ is chosen for the sake of simplicity) using rational functions. After a brief presentation of the general Hermite problem, we investigate two cases.

In the first one, f and f' are given and it is proved that for any monotonic set of data, it is always possible to construct a monotonic rational function of type [3/2] interpolating those data. Positive or convex interpolants can be build by similar algorithms. This first problem has been studied by many people, in particular by Gregory and coworkers in several papers, e.g. [8, 11]. Even if the results are similar, our approach is slightly different as it is based on the properties of a control polygon associated with the function.

The control polygon is again used as a tool in the second case where f, f' and f'' are given. For any positive (resp. monotonic, convex) set of data, algorithms are designed to construct a positive (resp. monotonic, convex) rational function of type [5/4] interpolating any suitable data. This second problem had not been considered in its full generality.

In the two cases, the interpolant is depending on a free parameter σ and for each subcase, we give an algorithm to obtain the solution and not only a result of existence. Error estimates are set up and numerical examples illustrate the various algorithms adapted to each problem.

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We only cite some papers solving this problem by using various families of methods. They can be classified as follows: polynomials with variable degrees [4], Chebyshev systems [6], rational functions [2, 3, 8, 11], polynomial splines [7, 9, 18] and subdivision methods [12, 14, 15]. A different approach for C^2 (and more) interpolation with shape constrains was proposed in [5, 13] with parametric curves.

Here is an outline of the paper. In Section 2, we study some properties of the families $\mathcal{R}[n/n-1]$ of rational functions that are used later. For this set, we have the possibility of associating a control polygon to any function $R \in \mathcal{R}[n/n-1]$ which is obtained by reproduction of affine polynomials (this idea was introduced in [16, 17]). As usual in CAGD, thanks to the total positivity of basic functions, the shape properties of the control polygon imply those of the underlying function (see e.g. [10]). In Section 3, we give the general solutions of various Hermite interpolation problems at endpoints of the interval $[0, 1]$ by rational functions in convenient spaces $\mathcal{R}[n/n-1]$.

Sections 4 and 5 are devoted to the algorithms for shape constrained problems. As the considered problems have an infinite number of solutions, we deliberately chose the parameters in order to design simple algorithms in the forms of [12] for the computation of satisfying solutions. Then numerical examples test the feasability of the methods. We have only proposed a brief approach of the case $\mathcal{R}[3/2]$ (already studied in [8]) with the solution of the monotonic problem deduced from the shape of the control polygon. For the second Hermite interpolation problem in $\mathcal{R}[5/4]$, the study is subdivided into three parts corresponding to constraints of positivity, monotonicity and convexity.

Finally, in the last section, first subsection, we give some bounds on the errors $f - R$ and $f' - R'$ for functions f having a bounded fourth (resp. sixth) derivative. In addition, we explicitly compute some values of the constants involved in majorations for both families $\mathcal{R}[3/2]$ and $\mathcal{R}[5/4]$ of rational interpolants. For $f - R$, those constants can be bounded independently from the parameter introduced in the algorithms allowing shape constrains. Then, in the second subsection, we show that, for Hermite interpolation in $\mathcal{R}[5/4]$ on the interval $[0, h]$, we can obtain an error $f-R$ in $O(h^4)$ (instead of $O(h^2)$), with our choice of the parameter σ in the monotonic case with data such that $f'(a) > 0$ and $f'(a + h) > 0$. A similar result was already obtained in [8] for interpolants in $\mathcal{R}[3/2]$.

It is clear that the local schemes presented in this paper can be used for piecewise Hermite interpolation with shape constraints, as done e.g. in [12]. In addition, the latter may vary in each interval, shape constraints can be accumulated and the choice of the local rational interpolant can be adapted to each specific case.

2 Basis and Control Polygon

For every positive integer $n \geq 2$, a rational function of type $\mathcal{R}[n/n-1]$ is defined by

$$R(t) = \frac{P(t)}{Q(t)} = \frac{\sum_{i=0}^{n} \bar{w}_i c_i B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}, \tag{1}$$

where for $p = n-1$ or $p = n$, the $B_i^p(t) = \binom{p}{i} t^i (1-t)^{p-i}$, $i = 0, \ldots, p$ are the Bernstein polynomials in $\mathbb{P}_p$. For the sake of convenience, we set $B_i^p(t) = 0$ for $i < 0$ and $i > p$.

The weights $\{w_j\}_{j=0,\ldots,n-1}$ of the denominator are supposed to be positive and will be the *shape parameters*. The weights $\{\bar{w}_i\}_{i=0,\ldots,n}$ of the numerator are depending on the w_j's and are chosen in such a way that that both monomials $m_0(x) = 1$ and $m_1(x) = x$ have a rational representation of the above type (1). Following [17], these weights are computed as in Proposition 1 below. The c_i's are the *control coefficients* of the rational function R.

Definition 1 *A function f is* reproduced *by the representation of type $\mathcal{R}[n/n-1]$ if there exists*

$(\xi_i)_{i=0,\dots,n}$ *such that for* $t \in [0,1]$, $f(t) = \dfrac{\sum_{i=0}^{n} \bar{w}_i f(\xi_i) B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}$.

Proposition 1 *Both monomials* m_0, m_1 *are reproduced by the rational functions of type* $\mathcal{R}[n/n-1]$ *if and only if, for* $i = 0, \dots, n$,

$$\bar{w}_i = \frac{i}{n} w_{i-1} + \left(1 - \frac{i}{n}\right) w_i, \quad (2)$$

$$\xi_i = \frac{i}{n} \frac{w_{i-1}}{\bar{w}_i} \quad (3)$$

where $w_{-1} = w_n = 0$.

Proof: Since for any n, i, $j \in \mathbb{N}$ and $t \in [0,1]$, we have $jB_j^n(t) = ntB_{j-1}^{n-1}(t)$ and $(n-j)B_j^n(t) = n(1-t)B_j^{n-1}(t)$, it is easy to prove that when defining $\bar{w}_i$ by (2) with $w_{-1} = w_n = 0$, we obtain by degree raising

$$1 \times \sum_{j=0}^{n-1} w_j B_j^{n-1}(t) = \sum_{i=0}^{n} \bar{w}_i B_i^n(t),\ t \in [0,1].$$

Conversely, the unicity of the decomposition of $1 \times \sum_{j=0}^{n-1} w_j B_j^{n-1}(t)$ in the Bernstein basis $\{B_i^n(t)\}_{i=0,\dots,n}$ gives the unicity of the $\bar{w}_i$ depending on the w_j.

Now, in a similar way, if we choose ξ_i defined by (3) for $i = 0, \dots, n$, we have

$$t \times \sum_{j=0}^{n-1} w_j B_j^{n-1}(t) = \sum_{i=0}^{n} \bar{w}_i \xi_i B_i^n(t) \Leftrightarrow m_1(t) = \frac{\sum_{i=0}^{n} \bar{w}_i m_1(\xi_i) B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}.$$

Again, we have unicity. $\square$

Proposition 2 *For* $i = 0, \dots, n$, *let us define* $\rho_i(t) = \dfrac{\bar{w}_i B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}$. *Then* $(\rho_0, \dots, \rho_n)$ *is totally positive on* $[0,1]$.

Proof: It is well known that the Bernstein basis is totally positive on $[0,1]$. Following [10], if $(\phi_0, \dots, \phi_d)$ is totally positive on I and if f is a positive function, then, $(f\phi_0, \dots, f\phi_d)$ is also totally positive on I. When choosing $f(t) = \dfrac{1}{\sum_{j=0}^{n-1} w_j B_j^{n-1}}$, we obtain that $\left(\dfrac{B_i^n}{\sum_{j=0}^{n-1} w_j B_j^{n-1}}\right)_{i=0,\dots,n}$ is totally positive. Also, if $A \in \mathbb{R}^{(d+1)\times(d+1)}$ is a totally positive matrix and $(\phi_0, \dots, \phi_d)$ is totally positive on I then $(\psi_0, \dots, \psi_d)$ is totally positive on I where $\psi_i = \sum_{j=0}^{d} a_{ij} \phi_j$. If A is the diagonal matrix with $(\bar{w}_0, \dots, \bar{w}_n)$ on the diagonal, then $(\rho_0, \dots, \rho_n)$ is *totally positive on* $[0,1]$. $\square$

Definition 2 *Let* $R(t) = \dfrac{\sum_{i=0}^{n} \bar{w}_i c_i B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}$ *be a rational function of type* $\mathcal{R}[n/n-1]$ *satisfying* (2). *The* control polygon *of* $R(t)$ *is the broken line with control vertices* $\{A_i = (\xi_i, c_i)\}_{i=0,\dots,n}$ *where* ξ_i *is defined by* (3).

A fundamental and useful property of the total positivity is that

Proposition 3 *The curve $R(t)$ inherits the shape properties (positivity, monotonicity, convexity) of its control polygon.*

The proof can be found in many publications, for example [10].

3 Hermite Interpolation

We recall that for a sequence $(x_i)_{i=0,\ldots}$, $\Delta x_i = x_{i+1} - x_i$ and for $j > 1$, $\Delta^j x_i = \Delta^{j-1} x_i - \Delta^{j-1} x_i$. Conversely, by a recursion, it holds

$$x_k = \sum_{\ell=0}^{k} \binom{k}{\ell} \Delta^\ell x_0, \quad k = 0, 1, \ldots. \tag{4}$$

Proposition 4 *Let* $R(t) = \dfrac{\sum_{i=0}^{n} \bar{w}_i c_i B_i^n(t)}{\sum_{j=0}^{n-1} w_j B_j^{n-1}(t)}$. *Then*

$$c_k = \frac{1}{\bar{w}_k} \sum_{i=0}^{k} \left[\sum_{j=0}^{k-i} \frac{\binom{k}{j+i}\binom{n-1}{j}}{\binom{n}{j+i}} \Delta^j w_0 \right] \frac{R^{(i)}(0)}{i!}, \quad k = 0, \ldots, n. \tag{5}$$

Proof: Using well known results on the differentiation of Bézier curves, for $m \leq p$, we get $\dfrac{d^m}{dt^m} \left(\sum_{\ell=0}^{p} z_\ell B_\ell^p \right)(t) = p(p-1)\ldots(p-m+1) \sum_{\ell=0}^{p-m} \Delta^m z_\ell B_\ell^{p-m}(t)$.
At $t = 0$, this gives $\dfrac{d^m}{dt^m} \left(\sum_{\ell=0}^{p} z_\ell B_\ell^p \right)(0) = m! \dbinom{p}{m} \Delta^m z_0$.

When computing a Taylor expansion of $R(t) \times \sum_{j=0}^{n-1} w_j B_j^{n-1}(t) = \sum_{k=0}^{n} \bar{w}_k c_k B_k^n(t)$, at $t = 0$, we obtain

$$\left(\sum_{i=0}^{n} \frac{1}{i!} R^{(i)}(0) t^i + o(t^n) \right) \left(\sum_{j=0}^{n-1} \binom{n-1}{j} \Delta^j w_0 t^j \right) = \sum_{\ell=0}^{n} \binom{n}{\ell} \Delta^\ell (\bar{w}_0 c_0) t^\ell.$$

The coefficients of the monomial t^ℓ give for $\ell \leq n$,

$$\binom{n}{\ell} \Delta^\ell (\bar{w}_0 c_0) = \sum_{i=0}^{\ell} \frac{1}{i!} \binom{n-1}{\ell-i} \Delta^{\ell-i} w_0 R^{(i)}(0).$$

Now, using (4), we obtain that for $k \leq n$,

$$\begin{aligned} \bar{w}_k c_k &= \sum_{\ell=0}^{k} \binom{k}{\ell} \Delta^\ell (\bar{w}_0 c_0) = \sum_{\ell=0}^{k} \frac{\binom{k}{\ell}}{\binom{n}{\ell}} \sum_{i=0}^{\ell} \frac{\binom{n-1}{\ell-i}}{i!} \Delta^{\ell-i} w_0 R^{(i)}(0) \\ &= \sum_{i=0}^{k} \left[\sum_{j=0}^{k-i} \frac{\binom{k}{j+i}\binom{n-1}{j}}{\binom{n}{j+i}} \Delta^j w_0 \right] \frac{1}{i!} R^{(i)}(0) \end{aligned}$$

which gives the result. □

Now we are ready to interpolate a function and its derivatives at the endpoints of the interval $I = [0, 1]$.

Proposition 5 *Given Hermite data* $\left(r_0^{(i)}\right)_{i=0,\ldots,p}$, $\left(r_1^{(j)}\right)_{j=0,\ldots,q}$ *and* $n = p+q+1$, *then the rational function* $R(t)$ *defined by* (1) *satisfies* $R^{(i)}(0) = r_0^{(i)}$ *for* $i = 0,\ldots,p$ *and* $R^{(j)}(1) = r_1^{(j)}$ *for* $j = 0,\ldots,q$ *provided that the control coefficients are defined by*

$$\begin{aligned} c_k &= \frac{1}{\bar{w}_k}\sum_{i=0}^{k}\left[\sum_{j=0}^{k-i}\frac{\binom{k}{j+i}\binom{n-1}{j}}{\binom{n}{j+i}}\Delta^j w_0\right]\frac{r_0^{(i)}}{i!},\ k = 0,\ldots,p,\\ c_{n-\ell} &= \frac{1}{\bar{w}_{n-\ell}}\sum_{i=0}^{\ell}\left[\sum_{j=0}^{\ell-i}\frac{\binom{\ell}{j+i}\binom{n-1}{j}}{\binom{n}{j+i}}(-1)^j\Delta^j w_{n-1-j}\right]\frac{(-1)^i r_1^{(i)}}{i!},\ \ell = 0,\ldots,q. \end{aligned}$$

Proof: The first part is a direct consequence of the previous Proposition. For the second part, we use the symmetry derived from changing t into $1-t$. □

Corollary 6 *When adding the hypotheses* (2) *and* (3), *for* n *large enough, this gives for the first and last coefficients:*

$$\begin{aligned} c_0 &= r_0^{(0)},\\ c_1 &= r_0^{(0)} + \xi_1 r_0^{(1)},\\ c_2 &= r_0^{(0)} + \xi_2\left(r_0^{(1)} + \frac{w_0}{2(n-1)w_1}r_0^{(2)}\right),\\ c_3 &= r_0^{(0)} + \xi_3\left(r_0^{(1)} + \frac{-w_0+(n-1)w_1}{(n-1)(n-2)w_2}r_0^{(2)} + \frac{w_0}{3(n-1)(n-2)w_2}r_0^{(3)}\right), \end{aligned}$$

and

$$\begin{aligned} c_n &= r_1^{(0)},\\ c_{n-1} &= r_1^{(0)} + (\xi_{n-1}-1)r_1^{(1)},\\ c_{n-2} &= r_1^{(0)} + (\xi_{n-2}-1)\left(r_1^{(1)} - \frac{w_{n-1}}{2(n-1)w_{n-2}}r_1^{(2)}\right),\\ c_{n-3} &= r_1^{(0)} + (\xi_{n-3}-1)\left(r_1^{(1)} - \frac{-w_{n-1}+(n-1)w_{n-2}}{(n-1)(n-2)w_{n-3}}r_1^{(2)} + \frac{w_{n-1}}{3(n-1)(n-2)w_{n-3}}r_1^{(3)}\right). \end{aligned}$$

As the interpolation schemes described above reproduce affine polynomials (proposition 1), then classical results in Approximation Theory imply the following

Proposition 7 *For* $n \geq 1$, *if* R *defined by* (1) *with* (2) *and* (3) *is a Hermite interpolant of* f *at the enpoints of* $[0, 1]$ *(i.e. conclusion of Proposition 5), then the following error bounds hold:*

$$|R - f|_\infty \leq C_0|f''|_\infty, \quad |R' - f'|_\infty \leq C_1|f''|_\infty$$

where the constants depend, a priori, on the interpolation scheme.

Remark 1. *On an interval of length* h, *the previous proposition gives respectively* $|R - f|_\infty \leq C_0 h^2|f''|_\infty$ *and* $|R' - f'|_\infty \leq C_1 h|f''|_\infty$.

Following a technique described in [8], these results are improved in Sections 5 below.

4 C^1-Interpolation with Shape Constraints

We only study the case of monotonicity but algorihtms can be given for other cases such as positivity or convexity. The monotonicity of the interpolant has already been studied in [8]. However, as it is completely determined by the monotonicity of the initial control polygon, we present a shorter proof of the results. If f is decreasing then $-f$ is increasing so that we can restrict our discussion to increasing interpolants.

Given 4 data $\left(r_0^{(i)}\right)_{i=0,1}$ and $\left(r_1^{(j)}\right)_{j=0,1}$, which fit the the shape constraints of monotonicity, we want to construct a rational interpolant on the interval $[0,1]$ having the same properties.

We choose $n = 3$, $w_0 = w_2 = 1$ and $w_1 = \dfrac{\sigma - 1}{2}$, where σ is a free parameter satisfying $\sigma \geq 3$. From (2) and (3), we deduce that

$$\begin{aligned}
&\bar{w}_0 = \bar{w}_3 = 1, \qquad \bar{w}_1 = \bar{w}_2 = \frac{\sigma}{3} \\
&\xi_0 = 0, \quad \xi_1 = \frac{1}{\sigma}, \quad \xi_2 = 1 - \xi_1 = \frac{\sigma - 1}{\sigma}, \quad \xi_3 = 1 - \xi_0 = 1.
\end{aligned}$$

We have $0 = \xi_0 < \xi_1 \leq \dfrac{1}{3} < \dfrac{2}{3} \leq \xi_2 < \xi_3 = 1$, and we have seen in Corollary 6 that

$$\begin{aligned}
c_0 = r_0^{(0)}, \quad c_1 = r_0^{(0)} + \xi_1 r_0^{(1)} = r_0^{(0)} + \frac{r_0^{(1)}}{\sigma}, \\
c_2 = r_1^{(0)} + (\xi_2 - 1) r_1^{(1)} = r_1^{(0)} - \frac{r_1^{(1)}}{\sigma}, \quad c_3 = r_1^{(0)}.
\end{aligned} \tag{6}$$

Monotone C^1-Interpolants:

For the design of an algorithm constructing a nondecreasing interpolant we assume that

$$r_0^{(0)} \leq r_1^{(0)}, \quad r_0^{(1)} \geq 0,\ r_1^{(1)} \geq 0 \text{ and } r_0^{(1)} = r_1^{(1)} = 0 \text{ if } r_0^{(0)} = r_1^{(0)}. \tag{7}$$

In the latter case the interpolant is constant.

Then, for $\sigma \geq 3$, we have

$$c_0 = r_0^{(0)} \leq c_1 = r_0^{(0)} + \xi_1 r_0^{(1)} \leq c_2 = r_1^{(0)} - \xi_1 r_1^{(1)} \leq c_3 = r_1^{(0)}$$

provided $c_2 - c_1 \geq 0$ or equivalently

$$\sigma \left(r_1^{(0)} - r_0^{(0)}\right) \geq r_1^{(1)} + r_0^{(1)}. \tag{8}$$

If (7) holds then the following algorithm will compute a nondecreasing rational interpolant on $[0,1]$. It will be strictly increasing if $r_1^{(0)} > r_0^{(0)}$ and (8) holds with strict inequality.

Algorithm 8 (Nondecreasing Interpolant)

1. *$\sigma := 3$;*
2. *If $r_0^{(0)} < r_1^{(0)}$ then $\sigma := \max\left(\sigma, \dfrac{r_1^{(1)} + r_0^{(1)}}{r_1^{(0)} - r_0^{(0)}}\right)$*

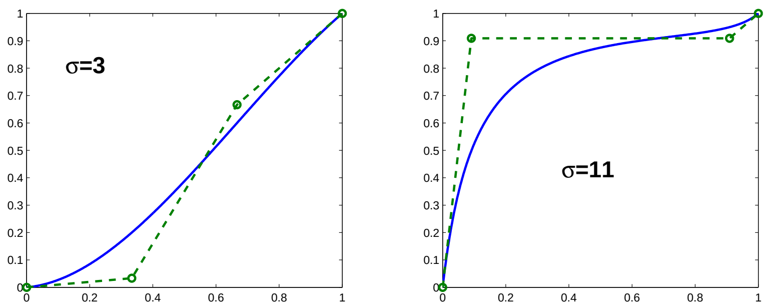

Figure 1: Non decreasing C^1–interpolants.

3. *Compute initial control coefficients* $(c_i)_{i=0,\dots,3}$ *using* (6).

4. *Compute* $R(t)$ *using* (1) *with* $w_0 = w_2 = 1$, $w_1 = \dfrac{\sigma - 1}{2}$ *and* $\bar{w}_0 = \bar{w}_3 = 1$, $\bar{w}_1 = \bar{w}_2 = \dfrac{\sigma}{3}$.

Note that if the initial control points are aligned, then the rational interpolant is the line segment connecting the first and last control point.

Example 1 *In Figure 1, we start with* $r_0^{(0)} = 0$, $r_1^{(0)} = 1$, $r_1^{(1)} = 1$, *then on the left* $r_0^{(1)} = 0.1$ *which gives* $\sigma = 3$, *and on the right* $r_0^{(1)} = 10$ *and* $\sigma = 11$.

5 C^2 interpolation with shape constraints

Given 6 data $\{r_0^{(i)}\}_{i=0,1,2}$ and $\{r_1^{(j)}\}_{j=0,1,2}$, which fit the shape constraints (positivity, monotonicity or convexity), we want to construct a rational interpolant on the interval $[0, 1]$ with the same shape property.

We choose $n = 5$, $w_0 = w_4 = 1$, $w_1 = w_3 = \dfrac{\sigma - 1}{4}$ and $w_2 = \dfrac{(\sigma - 1)(\sigma - 2)}{12}$, where $\sigma \geq 5$ is a free parameter. From (2) and (3), we deduce that

$$\bar{w}_0 = \bar{w}_5 = 1, \quad \bar{w}_1 = \bar{w}_4 = \sigma/5, \quad \bar{w}_2 = \bar{w}_3 = \frac{\sigma(\sigma - 1)}{20}$$
$$\xi_0 = 0, \quad \xi_1 = 1/\sigma, \quad \xi_2 = 2/\sigma,$$
$$\xi_3 = 1 - \xi_2 = \frac{\sigma - 2}{\sigma}, \ \xi_4 = 1 - \xi_1 = \frac{\sigma - 1}{\sigma}, \ \xi_5 = 1 - \xi_0 = 1.$$

We have $0 = \xi_0 < \xi_1 < \xi_2 \leq \dfrac{2}{5} < \dfrac{3}{5} \leq \xi_3 < \xi_4 < \xi_5 = 1$, and in order to get the Hermite interpolation, we deduce from Corollary 6 that

$$\begin{aligned} c_0 &= r_0^{(0)}, \quad c_1 = r_0^{(0)} + \frac{1}{\sigma} r_0^{(1)}, \quad c_2 = r_0^{(0)} + \frac{2}{\sigma} r_0^{(1)} + \frac{1}{\sigma(\sigma - 1)} r_0^{(2)}, \\ c_3 &= r_1^{(0)} - \frac{2}{\sigma} r_1^{(1)} + \frac{1}{\sigma(\sigma - 1)} r_1^{(2)}, \quad c_4 = r_1^{(0)} - \frac{1}{\sigma} r_1^{(1)}, \quad c_5 = r_1^{(0)}. \end{aligned} \tag{9}$$

5.1 Nonnegative C^2-Interpolants

The algorithm for constructing a nonnegative interpolant begins with the hypotheses

$$\begin{array}{lll} r_0^{(0)} \geq 0, & r_0^{(1)} \geq 0 \text{ if } r_0^{(0)} = 0, & r_0^{(2)} \geq 0 \text{ if } (r_0^{(0)} = 0 \;\&\; r_0^{(1)} = 0), \\ r_1^{(0)} \geq 0, & r_1^{(1)} \leq 0 \text{ if } r_1^{(0)} = 0, & r_1^{(2)} \geq 0 \text{ if } (r_1^{(0)} = 0 \;\&\; r_1^{(1)} = 0). \end{array} \tag{10}$$

Under these assumptions, nonnegative initial control coefficients $c_0, \ldots, c_5$ defined in (9) can always be obtained by choosing w sufficiently large.

Indeed, by the hypothesis (10), $c_0 = r_0^{(0)}$ and $c_5 = r_1^{(0)}$ are non negative. We have to ensure that $c_i \geq 0$ for $i = 1, 2, 3, 4$ which is equivalent to

$$\sigma r_0^{(0)} + r_0^{(1)} \geq 0, \tag{11}$$

$$\sigma(\sigma-1) r_0^{(0)} + 2(\sigma-1) r_0^{(1)} + r_0^{(2)} \geq 0, \tag{12}$$

$$\sigma(\sigma-1) r_1^{(0)} - 2(\sigma-1) r_1^{(1)} + r_0^{(2)} \geq 0, \tag{13}$$

$$\sigma r_1^{(0)} - r_0^{(1)} \geq 0, \tag{14}$$

Since $r_0^{(0)} \geq 0$ and $\sigma \geq 5$, we obtain that $\sigma(\sigma-1) r_0^{(0)} \geq (\sigma-1)^2 r_0^{(0)}$ and a similar result for $r_1^{(0)}$. Thus (12) and (13) are satisfied whenever

$$(\sigma-1)^2 r_0^{(0)} + 2(\sigma-1) r_0^{(1)} + r_0^{(2)} \geq 0, \tag{15}$$

$$(\sigma-1)^2 r_1^{(0)} - 2(\sigma-1) r_1^{(1)} + r_0^{(2)} \geq 0. \tag{16}$$

We notice that (15) and (16) are quadratic polynomials in the variable $\sigma - 1$.

- If $r_0^{(0)} = 0$ and $r_0^{(1)} = 0$, then $r_0^{(2)} \geq 0$ so that (11) and (15) are satisfied with $\sigma = 5$,
- If $r_0^{(0)} = 0$ and $r_0^{(1)} > 0$, then (11) is satisfied for $\sigma = 5$ and (15) is satisfied as soon as $\sigma \geq 1 - \dfrac{r_0^{(2)}}{2 r_0^{(1)}}$,
- If $r_0^{(0)} > 0$, then (11) is satisfied for $\sigma \geq -\dfrac{r_0^{(1)}}{r_0^{(0)}}$. Then we compute $\delta_0 = \left(r_0^{(1)}\right)^2 - r_0^{(0)} r_0^{(2)}$. If $\delta_0 \leq 0$, then (15) is satisfied for $\sigma = 5$ and if $\delta_0 > 0$, then (15) is satisfied for $\sigma \geq 1 + \dfrac{-r_0^{(1)} + \sqrt{\delta_0}}{r_0^{(0)}}$.

The conditions at the other end of the interval can be studied in a similar way.

If (10) holds then the following algorithm will compute a nonnegative interpolant on $[0, 1]$.

Algorithm 9 (Nonnegative C^2-Interpolant)

1. *Compute* σ
   - *(a)* $\sigma := 5$*;*
   - *(b)* **If** $r_0^{(0)} = 0$ **Then** *If* $r_0^{(1)} > 0$ *Then* $\sigma := \max\left(\sigma, 1 - \dfrac{r_0^{(2)}}{2 r_0^{(1)}}\right)$,

     Else Then

i. $\delta_0 := \left(r_0^{(1)}\right)^2 - r_0^{(0)} r_0^{(2)}$,

ii. If $\delta_0 \le 0$ *Then* $\sigma := \max\left(\sigma, -\dfrac{r_0^{(1)}}{r_0^{(0)}}\right)$,

Else Then $\sigma := \max\left(\sigma, -\dfrac{r_0^{(1)}}{r_0^{(0)}}, 1 + \dfrac{-r_0^{(1)} + \sqrt{\delta_0}}{r_0^{(0)}}\right)$;

(c) **If** $r_1^{(0)} = 0$ **Then** *If* $r_1^{(1)} < 0$ *Then* $\sigma := \max\left(\sigma, 1 + \dfrac{r_1^{(2)}}{2r_1^{(1)}}\right)$,

Else Then

i. $\delta_1 := \left(r_1^{(1)}\right)^2 - r_1^{(0)} r_1^{(2)}$.

ii. If $\delta_1 \le 0$ *Then* $\sigma := \max\left(\sigma, +\dfrac{r_1^{(1)}}{r_1^{(0)}}\right)$,

Else Then $\sigma := \max\left(\sigma, +\dfrac{r_1^{(1)}}{r_1^{(0)}}, 1 + \dfrac{+r_1^{(1)} + \sqrt{\delta_1}}{r_1^{(0)}}\right)$;

2. Compute initial control coefficients $(c_i)_{i=0,\dots,5}$ *using* (9);

3. Compute $R(t)$ *using* $n = 5$, (2) *with* $w_0 = w_4 = 1$, $w_1 = w_3 = \dfrac{\sigma - 1}{4}$, $w_2 = \dfrac{(\sigma-1)(\sigma-2)}{12}$ *and* $\bar{w}_0 = \bar{w}_5 = 1$, $\bar{w}_1 = \bar{w}_4 = \dfrac{\sigma}{5}$, $\bar{w}_2 = \bar{w}_3 = \dfrac{\sigma(\sigma-1)}{20}$.

Example 2 *In Figure 2, we start with* $r_0^{(0)} = r_1^{(0)} = 1$, $r_1^{(1)} = -1$, $r_1^{(2)} = 0$ *then*

on the upper left, $\begin{cases} r_0^{(1)} = -1 \\ r_0^{(2)} = 5 \\ \sigma = 5 \end{cases}$, *on the upper right,* $\begin{cases} r_0^{(1)} = -5 \\ r_0^{(2)} = 5 \\ \sigma = 10.4721 \end{cases}$

on the lower left, $\begin{cases} r_0^{(1)} = -5 \\ r_0^{(2)} = 50 \\ \sigma = 5 \end{cases}$, *on the lower right,* $\begin{cases} r_0^{(1)} = -5 \\ r_0^{(2)} = -5 \\ \sigma = 14.6603 \end{cases}$

5.2 Monotone C^2-Interpolants

We assume that the Hermite data satisfy

$$\begin{array}{c} r_0^{(0)} \le r_1^{(0)}, \quad r_0^{(1)} \ge 0, \quad r_1^{(1)} \ge 0 \\ r_0^{(2)} \ge 0 \text{ if } r_0^{(1)} = 0, \quad r_1^{(2)} \le 0 \text{ if } r_1^{(1)} = 0, \quad r_0^{(1)} = r_1^{(1)} = r_0^{(2)} = r_1^{(2)} = 0 \text{ if } r_0^{(0)} = r_1^{(0)}. \end{array} \tag{17}$$

In the latter case the interpolant is constant.

Under these assumptions, nondecreasing initial control coefficients $c_0, \dots, c_5$ defined in (9) can always be obtained by choosing σ sufficiently large.

Indeed, for $\sigma \ge 5$, when computing (9), by the hypothesis (17), $c_1 - c_0 = \dfrac{1}{\sigma} r_0^{(1)}$ and $c_5 - c_4 = \dfrac{1}{\sigma} r_1^{(1)}$ are non negative. We have then to ensure that $c_i - c_{i-1} \ge 0$ for $i = 2, 3, 4$ which is equivalent to

$$(\sigma - 1) r_0^{(1)} + r_0^{(2)} \ge 0, \tag{18}$$

$$\sigma(\sigma-1)\left(r_1^{(0)} - r_0^{(0)}\right) - 2(\sigma - 1)\left(r_0^{(1)} + r_1^{(1)}\right) + r_1^{(2)} - r_0^{(2)} \ge 0, \tag{19}$$

$$(\sigma - 1) r_1^{(1)} - r_0^{(2)} \ge 0, \tag{20}$$

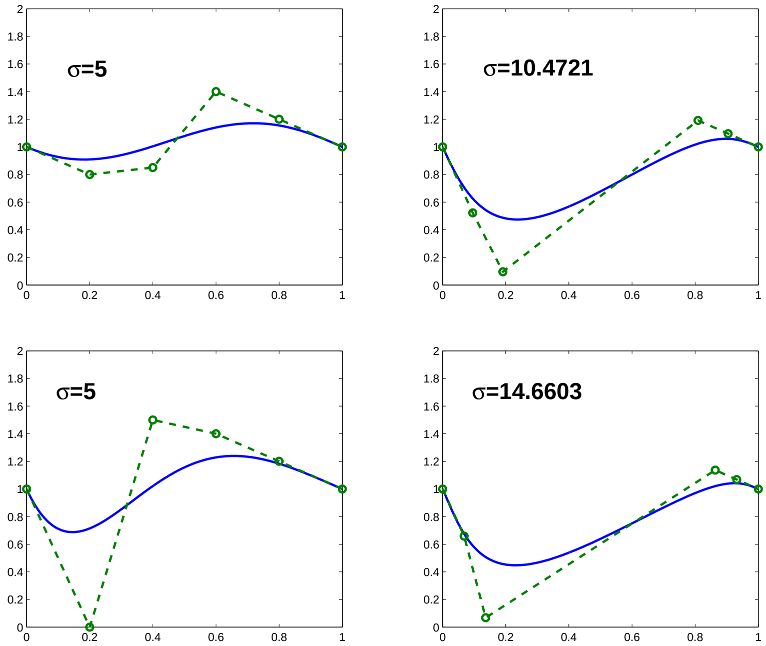

Figure 2: Non negative C^2–interpolants.

Since $r_1^{(0)} - r_0^{(0)} \geq 0$ and $\sigma \geq 5$, we obtain that (19) is satisfied whenever

$$(\sigma-1)^2\left(r_1^{(0)} - r_0^{(0)}\right) - 2(\sigma-1)\left(r_0^{(1)} + r_1^{(1)}\right) + r_1^{(2)} - r_0^{(2)} \geq 0 \tag{21}$$

which is again a quadratic polynomial in the variable $\sigma - 1$.

- If $r_0^{(1)} = 0$, then (18) is satisfied for $\sigma = 5$ since $r_0^{(2)} \geq 0$.
- if $r_0^{(1)} > 0$, then we choose $\sigma \geq 1 - \dfrac{r_0^{(2)}}{r_0^{(1)}}$.
- We have a similar computation for (20).
- If $r_0^{(0)} = r_1^{(0)}$, with (17) then (19) and (21) are trivially satisfied for $\sigma = 5$.
- If $r_0^{(0)} < r_1^{(0)}$, then we compute $\delta = \left(r_0^{(1)} + r_1^{(1)}\right)^2 - \left(r_1^{(0)} - r_0^{(0)}\right)\left(r_1^{(2)} - r_0^{(2)}\right)$. If $\delta \leq 0$, then (21) holds for $\sigma = 5$, and if $\delta > 0$, then (21) holds for $\sigma \geq 1 + \dfrac{r_0^{(1)} + r_1^{(1)} + \sqrt{\delta}}{r_1^{(0)} - r_0^{(0)}}$.

If (17) holds then the following algorithm will compute a nondecreasing rational interpolant on $[0, 1]$.

Algorithm 10 (Nondecreasing C^2-Interpolant)

1. Compute σ

(a) $\sigma := 5$*;*

(b) **If** $r_0^{(0)} < r_1^{(0)}$ **Then**

i. $\delta := \left(r_0^{(1)} + r_1^{(1)}\right)^2 - \left(r_1^{(0)} - r_0^{(0)}\right)\left(r_1^{(2)} - r_0^{(2)}\right)$,

ii. If $\delta > 0$ *Then* $\sigma := \max\left(\sigma, 1 + \dfrac{r_0^{(1)} + r_1^{(1)} + \sqrt{\delta}}{r_1^{(0)} - r_0^{(0)}}\right)$,

iii. If $r_0^{(1)} > 0$ *Then* $\sigma = \max\left(\sigma, 1 - \dfrac{r_0^{(2)}}{r_0^{(1)}}\right)$,

iv. If $r_1^{(1)} > 0$ *Then* $\sigma = \max\left(\sigma, 1 + \dfrac{r_1^{(2)}}{r_1^{(1)}}\right)$*;*

2. Compute initial control coefficients $(c_i)_{i=0,\dots,5}$ *using* (9)*;*

3. Compute $R(t)$ *using* $n = 5$*,* (2) *with* $w_0 = w_4 = 1$*,* $w_1 = w_3 = \dfrac{\sigma - 1}{4}$*,* $w_2 = \dfrac{(\sigma-1)(\sigma-2)}{12}$ *and* $\bar{w}_0 = \bar{w}_5 = 1$*,* $\bar{w}_1 = \bar{w}_4 = \dfrac{\sigma}{5}$*,* $\bar{w}_2 = \bar{w}_3 = \dfrac{\sigma(\sigma-1)}{20}$.

Example 3 *In Figure 3, we start with* $r_0^{(0)} = 0$*,* $r_1^{(0)} = 1$*,* $r_1^{(1)} = 1$*,* $r_1^{(2)} = -1$ *then*

on the upper left, $\left\{\begin{array}{c} r_0^{(1)} = 0.1 \\ r_0^{(2)} = 1 \\ \sigma = 5 \end{array}\right.$, *on the upper right,* $\left\{\begin{array}{c} r_0^{(1)} = 10 \\ r_0^{(2)} = 1 \\ \sigma = 23.0905 \end{array}\right.$

on the lower left, $\left\{\begin{array}{c} r_0^{(1)} = 0.1 \\ r_0^{(2)} = -1 \\ \sigma = 11 \end{array}\right.$, *on the lower right,* $\left\{\begin{array}{c} r_0^{(1)} = 10 \\ r_0^{(2)} = 10 \\ \sigma = 23.4891 \end{array}\right.$

5.3 Convex C^2-Interpolants

For this last ecase, we assume that the data satisfy

$$r_0^{(1)} < r_1^{(0)} - r_0^{(0)} < r_1^{(1)} \text{ and } r_0^{(2)} \geq 0,\, r_1^{(2)} \geq 0. \tag{22}$$

A convex control polygon can be built by choosing σ sufficiently large. With $\sigma \geq 5$, a condition to get such a polygon is $\dfrac{c_{i+1} - c_i}{\xi_{i+1} - \xi_i} \leq \dfrac{c_{i+2} - c_{i+1}}{\xi_{i+2} - \xi_{i+1}}$ for $i = 0, 1, 2, 3,$. This gives the following inequalities

$$r_0^{(2)} \geq 0, \tag{23}$$

$$\sigma\left(r_1^{(0)} - r_0^{(0)}\right) - 2\left(r_0^{(1)} + r_1^{(1)}\right) + \frac{r_1^{(2)} - r_0^{(2)}}{(\sigma - 1)} \geq (\sigma - 4) r_0^{(1)} + \frac{\sigma - 4}{\sigma - 1} r_0^{(2)}, \tag{24}$$

$$(\sigma - 4) r_1^{(1)} - \frac{\sigma - 4}{\sigma - 1} r_1^{(2)} \geq \sigma\left(r_1^{(0)} - r_0^{(0)}\right) - 2\left(r_0^{(1)} + r_1^{(1)}\right) + \frac{r_1^{(2)} - r_0^{(2)}}{(\sigma - 1)}, \tag{25}$$

$$r_1^{(2)} \geq 0. \tag{26}$$

The equations (23) and (26) are already in the hypothesis (22).

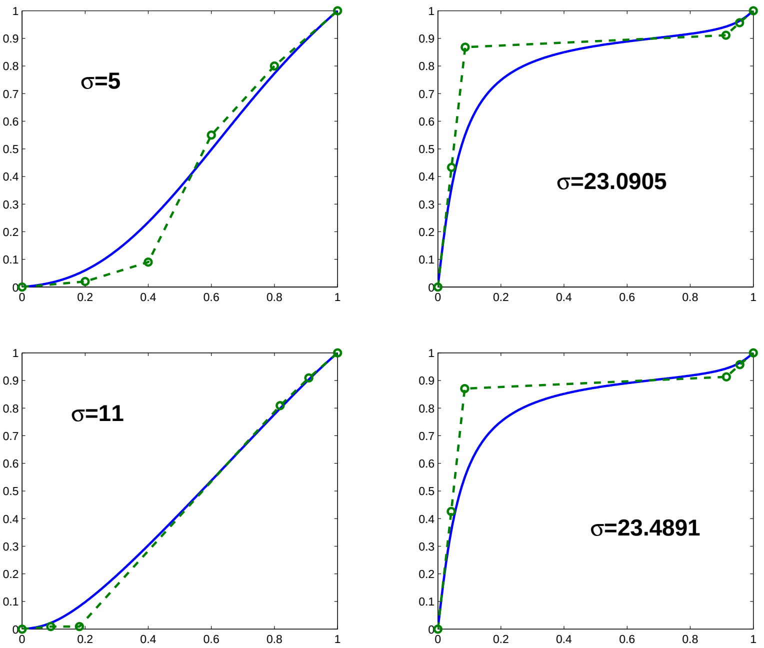

Figure 3: Non decreasing C^2–interpolants.

The equation (24) can be transformed into

$$
\begin{aligned}
& \sigma\left(r_1^{(0)}-r_0^{(0)}\right)-(\sigma-2)r_0^{(1)}-2r_1^{(1)}+\frac{1}{\sigma-1}\left(r_1^{(2)}-(\sigma-3)r_0^{(2)}\right)\geq 0 \\
\Leftrightarrow\quad & \sigma\left(r_1^{(0)}-r_0^{(0)}-r_0^{(1)}\right)+2\left(r_0^{(1)}-r_1^{(1)}-r_0^{(2)}/2\right)+\frac{1}{\sigma-1}\left(r_1^{(2)}+2r_0^{(2)}\right))\geq 0 \\
\Leftrightarrow\quad & \sigma(\sigma-1)\left(r_1^{(0)}-r_0^{(0)}-r_0^{(1)}\right)+2(\sigma-1)\left(r_0^{(1)}-r_1^{(1)}-r_0^{(2)}/2\right)+\left(r_1^{(2)}+2r_0^{(2)}\right)\geq 0
\end{aligned}
$$

Since $r_1^{(0)}-r_0^{(0)}-r_0^{(1)}>0$, a sufficient condition to satisfy this last inequality is

$$
(\sigma-1)^2\left(r_1^{(0)}-r_0^{(0)}-r_0^{(1)}\right)+2(\sigma-1)\left(r_0^{(1)}-r_1^{(1)}-r_0^{(2)}/2\right)+\left(r_1^{(2)}+r_0^{(2)}\right)\geq 0. \tag{27}
$$

Let

$$
\delta_0=\left(r_0^{(1)}-r_1^{(1)}-r_0^{(2)}/2\right)^2-\left(r_1^{(0)}-r_0^{(0)}-r_0^{(1)}\right)\left(r_1^{(2)}+2r_0^{(2)}\right).
$$

Either $\delta_0\leq 0$ and (27) is satisfied with $\sigma=5$, or $\delta_0>0$ and (27) is satisfied whenever $\sigma\geq 1+\dfrac{-r_0^{(1)}+r_1^{(1)}+r_0^{(2)}/2+\sqrt{\delta_0}}{r_1^{(0)}-r_0^{(0)}-r_0^{(1)}}$.

Similarly a sufficient condition to satisfy (25) is

$$
(\sigma-1)^2\left(r_1^{(1)}-r_1^{(0)}+r_0^{(0)}\right)+2(\sigma-1)\left(r_0^{(1)}-r_1^{(1)}-r_1^{(2)}/2\right)+\left(r_0^{(2)}+2r_1^{(2)}\right)\geq 0. \tag{28}
$$

Let

$$
\delta_1=\left(r_0^{(1)}-r_1^{(1)}-r_1^{(2)}/2\right)^2-\left(r_1^{(1)}-r_1^{(0)}+r_0^{(0)}\right)\left(r_0^{(2)}+2r_1^{(2)}\right).
$$

Either $\delta_1 \le 0$ and (28) is satisfied with $\sigma = 5$, or $\delta_1 > 0$ and (28) is satisfied whenever
$\sigma \ge 1 + \dfrac{-r_0^{(1)} + r_1^{(1)} + r_1^{(2)}/2 + \sqrt{\delta_1}}{r_1^{(1)} - r_1^{(0)} + r_0^{(0)}}$.

Under the assumption (22), the following algorithm will compute a convex interpolant.

Algorithm 11 (Convex C^2-Interpolant)

1. *$\sigma = 5$;*
2. $\delta_0 = \left(r_0^{(1)} - r_1^{(1)} - r_0^{(2)}/2\right)^2 - \left(r_1^{(0)} - r_0^{(0)} - r_0^{(1)}\right)\left(r_1^{(2)} + 2r_0^{(2)}\right)$;
3. *If* $\delta_0 > 0$ *Then* $\sigma = \max\left(\sigma, 1 + \dfrac{-r_0^{(1)} + r_1^{(1)} + r_0^{(2)}/2 + \sqrt{\delta_0}}{r_1^{(0)} - r_0^{(0)} - r_0^{(1)}}\right)$;
4. $\delta_1 = \left(r_0^{(1)} - r_1^{(1)} - r_1^{(2)}/2\right)^2 - \left(r_1^{(1)} - r_1^{(0)} + r_0^{(0)}\right)\left(r_0^{(2)} + 2r_1^{(2)}\right)$;
5. *If* $\delta_1 > 0$ *Then* $\sigma = \max\left(\sigma, 1 + \dfrac{-r_0^{(1)} + r_1^{(1)} + r_1^{(2)}/2 + \sqrt{\delta_1}}{r_1^{(1)} - r_1^{(0)} + r_0^{(0)}}\right)$;
6. *Compute initial control coefficients* $(c_i)_{i=0,\ldots,5}$ *using* (9).
7. *Compute* $R(t)$ *using* $n = 5$, (2) *with* $w_0 = w_4 = 1$, $w_1 = w_3 = \dfrac{\sigma - 1}{4}$, $w_2 = \dfrac{(\sigma-1)(\sigma-2)}{12}$ *and* $\bar{w}_0 = \bar{w}_5 = 1$, $\bar{w}_1 = \bar{w}_4 = \dfrac{\sigma}{5}$, $\bar{w}_2 = \bar{w}_3 = \dfrac{\sigma(\sigma-1)}{20}$.

Example 4 *In Figure 4, we start with* $r_0^{(0)} = r_1^{(0)} = 1$, $r_1^{(1)} = 4$, $r_1^{(2)} = 0$ *then*

on the upper left, $\begin{cases} r_0^{(1)} = -4 \\ r_0^{(2)} = 0 \\ \sigma = 5 \end{cases}$, *on the upper right,* $\begin{cases} r_0^{(1)} = -4 \\ r_0^{(2)} = 10 \\ \sigma = 6.6085 \end{cases}$

on the lower left, $\begin{cases} r_0^{(1)} = -1 \\ r_0^{(2)} = 0 \\ \sigma = 11 \end{cases}$, *on the lower right,* $\begin{cases} r_0^{(1)} = -1 \\ r_0^{(2)} = 10 \\ \sigma = 19.9443 \end{cases}$

6 Error estimates

In this section, we firstly study the errors $E(x) = f(x) - R(x)$ and $E'(x) = f'(x) - R'(x)$ on the interval $[0, 1]$, in the two previous cases $\mathcal{R}[3/2]$ and $\mathcal{R}[5/4]$. We complete the results of Proposition 7, by giving the majoration of $|E(x)|$ where the constants are independent of the parameter σ introduced in the previous sections with the counterpart of using higher derivatives of f. On an interval of length h, this give an error $E(x)$ in $O(h^2)$.

In a second part, we show that for the monotonic interpolant and data such that $f'(0) > 0$ and $f'(h) > 0$, instead of $O(h^2)$, we can obtain $O(h^4)$ by an appropriate choice of the parameter σ.

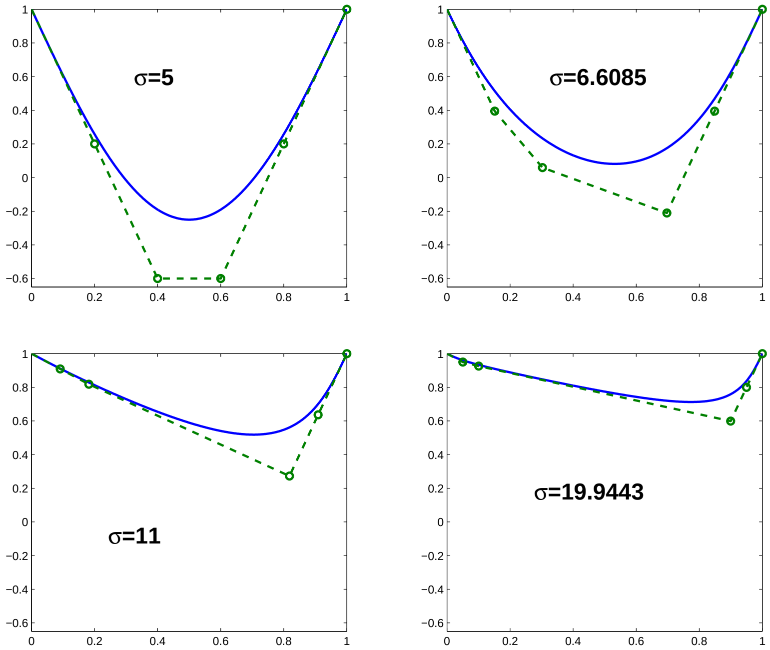

Figure 4: Convex C^2–interpolants.

6.1 Error on $[0, 1]$ for the parameters of Sections 4 and 5

We give the results and a summary of a proof. The first one is slightly better than the corresponding majoration given in [8]. They are obtained by using the fact that, in both cases, the numerator P of the rational interpolant $R = P/Q$ is equal to the Hermite interpolant of the function $g = Qf$. Therefore the error $E = f - P/Q$ can be written as $E = (g - P)/Q$ and the majorations follow from the classical majorations of errors in Hermite interpolation by polynomials.

Theorem 12 *Let f be a regular function on $[0, 1]$ and let R be the rational interpolant defined in the previous algorithms in $\mathcal{R}[3/2]$ (respectively in $\mathcal{R}[5/4]$). If $M_i = \max_{x\in[0,1]} |f^{(i)}(x)|$, then*

- *For $E_1(x) = f(x) - R(x)$ in $\mathcal{R}[3/2]$, if $\tau = \sigma - 3$, then*

$$|E_1(x)| \leq \tau \left(\frac{1}{32} M_2 + \frac{1}{96} M_3 + \frac{1}{1536} M_4 \right) + \frac{M_4}{384}. \tag{29}$$

- *Let $E_1'(x) = f'(x) - R'(x)$ in $\mathcal{R}[3/2]$, if $\tau = \sigma - 3$,*

$$|E_1'(x)| \leq \frac{\tau^2}{\tau + 4} \left(\frac{1}{8} M_2 + \frac{1}{24} M_3 + \frac{1}{384} M_4 \right) + \frac{3\tau + 8}{\tau + 4} \frac{M_4}{96}. \tag{30}$$

  The various constants can be majorized independently of τ:

$$\frac{\tau}{\tau + 4} \leq 1, \qquad \frac{3\tau + 8}{\tau + 4} \leq 3.$$

- *For $E_2(x) = f(x) - R(x)$ in $\mathcal{R}[5/4]$, if we set $\tau = \sigma - 5$ and $\mu = \frac{1}{2}(\tau + 3)$, then*

$$|E_2(x)| \leq \frac{\tau}{\tau\mu + 4\tau + 16}\left(\frac{\mu}{8}M_2 + \frac{\mu}{12}M_3 + \frac{2\mu+1}{192}M_4 + \frac{\mu+5}{2400}M_5 + \frac{\mu+4}{2880}\frac{M_6}{16}\right) + \frac{M_6}{2880}. \quad (31)$$

  With $q(\tau) = \tau\mu + 4\tau + 16$, the various constants can be majorized independently of τ

$$\frac{\tau\mu}{q(\tau)} \leq 1, \quad \frac{\tau(2\mu+1)}{q(\tau)} \leq 2, \quad \frac{\tau(\mu+5)}{q(\tau)} \leq \frac{5}{4}, \quad \frac{\tau(\mu+4)}{q(\tau)} \leq 1.$$

- *Let $E_2'(x) = f'(x) - R'(x)$ in $\mathcal{R}[5/4]$, then with a graphical help, we get*

$$\begin{aligned} |E_2'(x)| \quad &\leq \quad \tau\sqrt{5}\left(\frac{3\mu}{50}M_2 + \frac{\mu}{25}M_3 + \frac{2\mu+1}{400}M_4 + \frac{\mu+5}{5000}M_5 + \frac{\mu+4}{6000}\frac{M_6}{16}\right) \\ &+ \frac{1.03\tau^2}{\tau\mu + 4\tau + 16}\left(\frac{\mu}{8}M_2 + \frac{\mu}{12}M_3 + \frac{2\mu+1}{192}M_4 + \frac{\mu+5}{2400}M_5 + \frac{\mu+4}{2880}\frac{M_6}{16}\right) \quad (32) \\ &+ \left(\frac{1.03\tau}{2880} + \frac{\sqrt{5}}{6000}\right)M_6. \end{aligned}$$

Proof: We only give a summary of the proof of the third inequality.

The numerator of the interpolant $R = P/Q$ has the form

$$Q(x) = 1 + \tau X + \tau\mu X^2, \text{ where } X = x(1-x)$$

where are used the notations $\tau = \sigma - 5$ et $\mu = \frac{1}{2}(\sigma - 2) = \frac{1}{2}(\tau + 3)$. Let P be the quintic Hermite interpolant of the function $g = Qf$. According to [1] (chapter 2), the interpolation errors for g and its derivatives are respectively

$$\begin{aligned} g(x) - P(x) \quad &= \quad \frac{1}{720}X^3 D^6 g(c_0), && (33) \\ g'(x) - P'(x) \quad &= \quad \frac{1}{240}(1-2x)X^2 D^6 g(c_1), && (34) \\ g''(x) - P''(x) \quad &= \quad \frac{1}{120}X(1-5X)D^6 g(c_2), && (35) \end{aligned}$$

where c_0, c_1, c_2 are elements of $[0,1]$ depending on x.

First, one has

$$\begin{aligned} D^6 g(x) \quad = \quad & 360\tau\mu\, D^2 f(x) + 240\tau\mu(2x-1)D^3 f(x) + 30\tau(\mu - 1 - 6\mu X)D^4 f(x) \\ & -6\tau(2x-1)(1+2\mu X)D^5 f(x) + Q(x)D^6 f(x), \end{aligned}$$

then using the notations $M_k = \|f^{(k)}\|_\infty$ for $k \geq 1$ and the majorations

$$\frac{X^3}{Q(X)} \leq \frac{1}{4}\frac{1}{\tau\mu + 4\tau + 16}, \quad |\mu - 1 - 6\mu X| \leq \frac{1}{2} + \mu, \quad |2x-1|(1+2\mu X) \leq 1 + \mu/5$$

together with

$$|D^6 g(x)| \leq 360\tau\mu M_2 + 240\tau\mu M_3 + 15\tau(2\mu+1)M_4 + 6\tau(1+\mu/5)M_5 + (1 + \tau(\mu+4)/16)M_6$$

one finally gets

$$|E_2(x)| \leq \frac{\tau}{\tau\mu + 4\tau + 16}\left(\frac{\mu}{8}M_2 + \frac{\mu}{12}M_3 + \frac{2\mu+1}{192}M_4 + \frac{\mu+5}{2400}M_5 + \frac{\mu+4}{2880}\frac{M_6}{16}\right) + \frac{M_6}{2880} \quad (36)$$

Reminding that $\mu = \frac{1}{2}(\tau+3)$ and setting $q(\tau) = \tau\mu+4\tau+16$, the various constants can be majorized independently of τ

$$\frac{\tau\mu}{q(\tau)} \leq 1, \quad \frac{\tau(2\mu+1)}{q(\tau)} \leq 2, \quad \frac{\tau(\mu+5)}{q(\tau)} \leq \frac{5}{4}, \quad \frac{\tau(\mu+4)}{q(\tau)} \leq 1,$$

6.2 Parameters σ in $\mathcal{R}[5/4]$ giving errors in $O(h^4)$ on the interval $[0,h]$

In this subsection, for the monotonic interpolant and data such that $f'(0) > 0$ and $f'(h) > 0$, we show that, when interpolating the function in a small interval $[0,h]$, it is always possible to choose the parameter $\tau = \sigma - 5$ in such a way that $\tau = O(h^2)$. This implies, from the results of the preceding subsection, that the global majoration of the error $|E_2(x)|$ will be a $O(h^4)$.

The errors being studied for Hermite interpolation of f on $[0,h]$, the data are now

$$r_0^{(0)} = f(0), \quad r_0^{(1)} = f'(0), \quad r_0^{(2)} = f''(0),$$
$$r_1^{(0)} = f(h), \quad r_1^{(1)} = f'(h), \quad r_1^{(2)} = f''(h).$$

Thus, the B-coefficients of the rational interpolant R depending on the parameter σ are respectively (see (9) in Section 5):

$$c_0 = r_0^{(0)}, \quad c_1 = r_0^{(0)} + \frac{h}{\sigma} r_0^{(1)}, \quad c_2 = r_0^{(0)} + \frac{2h}{\sigma} r_0^{(1)} + \frac{h^2}{\sigma(\sigma-1)} r_0^{(2)},$$
$$c_3 = r_1^{(0)} - \frac{2h}{\sigma} r_1^{(1)} + \frac{h^2}{\sigma(\sigma-1)} r_1^{(2)}, \quad c_4 = r_1^{(0)} - \frac{h}{\sigma} r_1^{(1)}, \quad c_5 = r_1^{(0)}.$$

We set $\Delta^{(j)} = r_1^{(j)} - r_0^{(j)}$ for $j = 0,1,2$ and we also use $r_m^{(j)} = f^{(j)}(h/2)$, $j \geq 0$ for the derivatives at the midpoint of the interval. □

Theorem 13 (**Monotonicity**). *We assume that $f'(0) = r_0^{(1)} > 0$, $f'(h) = r_1^{(1)} > 0$ and $\Delta^{(0)} > 0$. Then, choosing as parameter $\tau^* = \sigma^* - 5$ with*

$$\sigma^* = \begin{cases} \max\left(5, 1 + \frac{h}{\Delta^{(0)}}(r_0^{(1)} + r_1^{(1)} + \sqrt{\delta})\right) & \text{if } \delta > 0, \\ 5 \text{ if } \delta \leq 0 \end{cases}$$

where $\delta = (r_0^{(1)} + r_1^{(1)})^2 - \Delta^{(0)}\Delta^{(2)}$, as in Algorithm 10 of Section 5, then we get

$$\tau^* = \max\left(0, \frac{h^2}{12}\frac{r_m^{(3)}}{r_m^{(1)}} + O(h^4)\right).$$

Proof: According to Subsection 5.2, sufficient conditions (18), (19), (20) on the parameter for R to be increasing are transformed into

$$(\sigma-1)r_0^{(1)} + hr_0^{(2)} \geq 0, \tag{37}$$
$$\sigma(\sigma-1)\Delta^{(0)} - 2(\sigma-1)h\left(r_0^{(1)} + r_1^{(1)}\right) + h^2\Delta_0^{(2)} \geq 0, \tag{38}$$
$$(\sigma-1)r_1^{(1)} - hr_0^{(2)} \geq 0, \tag{39}$$

Since $\sigma-1>0$, $r_0^{(1)}>0$ and $r_1^{(1)}>0$, Equations (37) and (39) are satisfied if h is small enough. Setting $\rho=\sigma-1$, since $\Delta^{(0)}>0$, a sufficient condition to satisfy (38) is

$$\Delta^{(0)}\rho^2-2h\left(r_0^{(1)}+r_1^{(1)}\right)\rho+h^2\Delta^{(2)}\geq 0 \tag{40}$$

The discriminant is $h^2\delta$, with $\delta=\left(r_0^{(1)}+r_1^{(1)}\right)^2-\Delta^{(0)}\Delta^{(2)}$.

If $\delta<0$, (40) is satisfied for any value of ρ.

If $\delta\geq 0$ or $\omega=\dfrac{\Delta^{(0)}\Delta^{(2)}}{\left(r_0^{(1)}+r_1^{(1)}\right)^2}\leq 1$, then ρ must be greater or equal to the positive root, i.e.

$$\rho\geq\rho^*=\frac{h}{\Delta^{(0)}}\left(r_0^{(1)}+r_1^{(1)}+\sqrt{\delta}\right),\quad i.e.\quad \sigma\geq 1+\rho^*$$

Let $\tau=\sigma-5$. We suppose that $\sigma>5$ otherwise $\tau=0$. We must have

$$\tau\geq\tau^*=\rho^*-4=\frac{1}{\Delta^{(0)}}\left(h\big(r_0^{(1)}+r_1^{(1)}+\sqrt{\delta}\big)-4\Delta^{(0)}\right).$$

Using the Taylor expansions at the midpoint of the interval

$$\Delta^{(0)}=hr_m^{(1)}+\frac{h^3}{24}r_m^{(3)}+O(h^5),\quad r_0^{(1)}+r_1^{(1)}=2r_m^{(1)}+\frac{h^2}{4}r_m^{(3)}+O(h^4),\quad \Delta^{(2)}=hr_m^{(3)}+O(h^3),$$

we deduce $\omega=\dfrac{h^2}{4}\dfrac{r_m^{(3)}}{r_m^{(1)}}+O(h^4)$, so that $\sqrt{\delta}=(r_0^{(1)}+r_1^{(1)})(1-\omega)^{1/2}=2r_m^{(1)}+O(h^4)$.

Thus, the numerator of τ^* has the expansion $h\left(r_0^{(1)}+r_1^{(1)}+\sqrt{\delta}\right)-4\Delta^{(0)}=\dfrac{h^3}{12}r_m^{(3)}+O(h^5)$, and finally

$$\tau^*=\frac{h^2}{12}\frac{r_m^{(3)}+O(h^2)}{r_m^{(1)}+O(h^2)}=\frac{h^2}{12}\frac{r_m^{(3)}}{r_m^{(1)}}+O(h^4)$$

In other words, if we choose as parameter:

$$\sigma=\sigma^*=\max\left(5,1+\frac{h}{\Delta^{(0)}}(h(r_0^{(1)}+r_1^{(1)}+\sqrt{\delta})\right),$$

we will get $0\leq\tau=\sigma-5\leq O(h^2)$. Since $\mu=\dfrac{1}{2}(\tau+3)=\dfrac{3}{2}+O(h^2)$, the majoration of the error $|E_2(x)|=|f(x)-R(x)|$ will be at least in $O(h^4)$ (see (31) with $[0,1]$ replaced by $[0,h]$).

We notice that the value σ^* was the value of σ obtained by Algorithm 10 for an interval $[0,h]$ with h small enough. □

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