C^1 INTERPOLATORY SUBDIVISION WITH SHAPE CONSTRAINTS FOR CURVES

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Abstract. We derive two reformulations of the C^1 Hermite subdivision scheme introduced by Merrien. One where we separate computation of values and derivatives and one based of refinement of a control polygon. We show that the latter leads to a subdivision matrix which is totally positive. Based on this we give algorithms for constructing subdivision curves that preserve positivity, monotonicity, and convexity.

Key words. Interpolation, Subdivision, Corner Cutting, Total Positivity, Positivity, Monotonicity, Convexity.

AMS subject classifications. 65D05, 65D17

1. Introduction. Subdivision is a technique for creating a smooth curve or surface out of a sequence of successive refinements of polygons, or grids see [2]. Subdivision has found applications in areas such as geometric design [7],[18], and in computer games and animation [5]. We consider here the two point Hermite scheme, the HC^1-algorithm, introduced in [13]. We start with values and derivatives at the endpoint of an interval and then compute values and derivatives at the midpoint. Repeating this on each subinterval we obtain in the limit a function with a certain smoothness. The scheme depends on two parameters α and β and it has been shown that the limit function is C^1 for a range C of these parameters. For more references to Hermite subdivision see [6, 12, 14, 15].

The strong locality of the HC^1-algorithm was used in [15] to construct subdivision curves with shape constraints like positivity, monotonicity, and convexity. A notion of control points, control coefficients and a Bernstein basis for two subfamilies of the HC^1-interpolant were introduced in [17].

In this paper we continue the study of subdivision with shape constraints initiated in [15, 17]. Before detailing our results let us first describe the shape preserving subdivision process and give an example. Suppose we have values $y_1, \ldots, y_n$ and derivatives $y'_1, \ldots, y'_n$ at some abscissae $t_1 < t_2 < \cdots < t_n$. With each subinterval $[t_i, t_{i+1}]$ we associate parameters $(\alpha_i, \beta_i) \in C$ chosen so that the HC^1-interpolant using data $(y_i, y'_i, y_{i+1}, y'_{i+1})$ has the required shape on $[t_i, t_{i+1}]$. We then obtain a C^1-function on $[t_1, t_n]$. As an illustration consider the function in Figure 1.1.

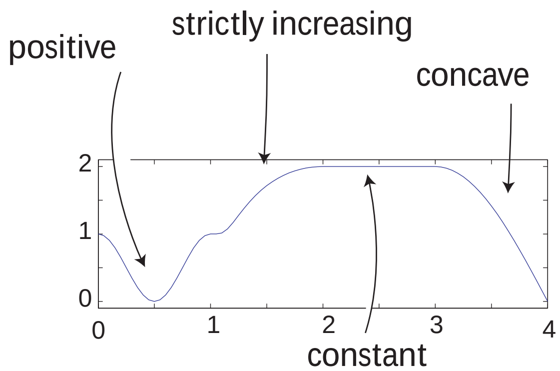

Fig. 1.1. *A given function.*

This function is defined on the interval $[0, 4]$. It is positive on $[0, 1]$, strictly increasing on $[1, 2]$, constant on $[2, 3]$ and concave on $[3, 4]$. Suppose we want to use subdivision to construct a C^1-

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approximation to this function with the same shape characteristics and that all we know about the function are the function values $y_1, \dots, y_n$ at some points $t_1 < \cdots < t_n$. We can achieve this with the HC^1-algorithm using only crude estimates for the derivatives $y'_1, \dots, y'_n$ as long as the transition points $1, 2, 3$ are among the abscissae and the chosen derivatives are consistent with the required shapes. See Section 6 for details. For classical curve based shape preserving algorithms we refer to [8, 10, 11] and references therein.

If we compare HC^1 to the Bezier or spline curves, we also introduce the control polygon. The originality of this family of interpolants is that monotonicity or convexity of the control polygon and of the function are equivalent (See Theorems 5.3 and 5.6). We observe that this only true in one direction for Bezier or spline curves.

Our paper can be detailed as follows. In Section 2, we recall the HC^1-algorithm and some properties which were proved in [15]. We give a new formulation of the HC^1-algorithm were we separate the computation of function values and derivatives. This formulation is useful for proving shape preserving properties and with the aid of this formulation we simplify the proofs of the main results in [15]. The new formulation also shows why the one parameter family given by $\alpha = \beta/(4(1-\beta))$ and $\beta \in [-1, 0)$ considered in [15, 17] really is an extension of the quadratic spline case. We will refer to this family as the *EQS-case* of the HC^1-algorithm. We also give a new domain C for C^1-convergence of the algorithm. In Section 3 we use the control points introduced in [17] to reformulate the HC^1 algorithm as a stationary subdivision algorithm called SC^1. The control points depend on a third parameter $\lambda \geq 2$ and we show convergence of the SC^1-algorithm for $(\alpha, \beta) \in C$ and $\lambda \geq 2$. Starting in Section 4, we restrict our attention to the EQS-case. By formulating the SC^1-algorithm as a corner cutting scheme we show that the subdivision matrix $\mathbf{S}$ is totally positive. We show this for an extended range of β and λ and also prove the total positivity of the HC^1-Bernstein basis. With this last property, the interpolant inherits shape properties of the control polygon such as nonnegativity, monotonicity or convexity. In Section 5, we give algorithms for interpolation with any of the previous shape constraints. An example based on Figure 1.1 is given in Section 6.

We also point out that Proposition 2.1 on one hand and Proposition 3.1 with Theorem 3.4 on the other hand show that we obtain two Lagrange subdivision schemes from the HC^1 Hermite subdivision scheme.

2. The HC^1 Algorithm . We recall the univariate version of the Hermite subdivision scheme for C^1 interpolation, given by Merrien [13] which we call here HC^1. We start with values $(f(a), p(a))$ and $(f(b), p(b))$ of a function f and of its first derivative $p = f'$ at the endpoints a, b of a bounded interval $I := [a, b]$ of $\mathbb{R}$. To build f and p on I, we proceed recursively. At step n ($n \geq 0$), let us denote by $\mathcal{P}_n$ the regular partition of I in 2^n subintervals and let us write $h_n := (b-a)/2^n$. If c and d are two consecutive points of $\mathcal{P}_n$, then we compute f and p at the midpoint $(c+d)/2$ according to the following scheme, which depends on two parameters α and β

$$\begin{aligned} f(\frac{c+d}{2}) &:= \frac{f(d)+f(c)}{2} + \alpha h_n [p(d) - p(c)], \\ p(\frac{c+d}{2}) &:= (1-\beta)\frac{f(d)-f(c)}{h_n} + \beta \frac{p(d)+p(c)}{2}. \end{aligned} \tag{2.1}$$

By applying these formulae on ever finer partitions, we define f and p on $\mathcal{P} = \cup \mathcal{P}_n$ which is a dense subset of I. We say that the scheme is *C^1-convergent* if, for any initial data, f and p can be extended from $\mathcal{P}$ to continuous functions on I with $p = f'$. We call f defined either on I or on $\mathcal{P}$ the *HC^1-interpolant* to the data.

The HC^1-algorithm can also be formulated as follows. We start with Hermite data f_0, p_0, f_1, p_1 at the endpoints of a finite interval $[a, b]$ and set $f_0^0 = f_0$, $p_0^0 = p_0$, $f_1^0 = f_1$, and $p_1^0 = p_1$. For

$n=0,1,2,\dots$, $h_n=2^{-n}(b-a)$, and $k=0,1,\dots,2^n-1$

$$f_{2k}^{n+1}:=f_k^n,\quad f_{2k+1}^{n+1}:=\frac{f_{k+1}^n+f_k^n}{2}+\alpha h_n\left(p_{k+1}^n-p_k^n\right), \tag{2.2}$$

$$p_{2k}^{n+1}:=p_k^n,\quad p_{2k+1}^{n+1}:=(1-\beta)\frac{f_{k+1}^n-f_k^n}{h_n}+\beta\frac{p_{k+1}^n+p_k^n}{2}, \tag{2.3}$$

and $f_{2^{n+1}}^{n+1}:=f_{2^n}^n$, $p_{2^{n+1}}^{n+1}:=p_{2^n}^n$. If the scheme is C^1-convergent with limit functions f and p then

$$f(t_k^n)=f_k^n,\ f'(t_k^n)=p(t_k^n)=p_k^n,\ t_k^n:=a+kh_n,\ k=0,1,\dots,2^n. \tag{2.4}$$

2.1. The Vector Space of HC^1-interpolants. To each choice of (α,β) there is a vector space

$$VC_{\alpha,\beta}^1(\mathcal{P}):=\{f:\mathcal{P}\to\mathbb{R}: f,p \text{ computed by } (2.2)-(2.4)\}$$

of HC^1-interpolants. If the scheme is C^1-convergent we define

$$VC_{\alpha,\beta}^1(I):=\{f:I\to\mathbb{R}: f|_{\mathcal{P}}\in VC_{\alpha,\beta}^1(\mathcal{P})\}.$$

The HC^1-*Hermite basis functions* $\{\phi_0,\psi_0,\phi_1,\psi_1\}$ are defined by taking as initial data the four unit vectors $e_j=(\delta_{i,j})_{i=1}^4$, respectively. They are always defined on $\mathcal{P}$ and the HC^1-interpolant corresponding to initial data (f_0,p_0,f_1,p_1) can be written $f=f_0\phi_0+p_0\psi_0+f_1\phi_1+p_1\psi_1$. Since the Hermite basis functions are clearly linearly independent on $\mathcal{P}$ they form a basis for $VC_{\alpha,\beta}^1(\mathcal{P})$. Thus $VC_{\alpha,\beta}^1(\mathcal{P})$ and $VC_{\alpha,\beta}^1(I)$ are vector spaces of dimension 4.

Let us denote the HC^1-interpolant to initial data sampled from a function g by $f=Hg$. By induction it is easy to see that for any (α,β) we have $g=Hg$ for all polynomials g of degree at most one, while $g=Hg$ for all quadratic polynomials if and only if $\alpha=-1/8$. We also have $g=Hg$ for all cubic polynomials if and only if $\alpha=-1/8$ and $\beta=-1/2$ and it can be shown that $x^k\neq Hx^k$ for any integer $k\geq 4$. The fact that the scheme reproduces polynomials up to a certain degree can be used to give error bounds, see [15, 5]. Assume (α,β) are chosen so that the scheme is C^1-convergent. Then there is a constant $C(\alpha,\beta)$ such that for all intervals $I=[a,b]$ and all $g\in C^k(I)$ we have

$$\|g-Hg\|_{L_\infty(I)}\leq C(\alpha,\beta)h^k\|g^{(k)}\|_{L_\infty(I)}, \tag{2.5}$$

where $h:=b-a$ and $k=2$ for most choices of α and β.

Notice some important choices of (α,β):

1. If $\alpha=-1/8$, $\beta=-1/2$, then f is the cubic polynomial known as the Hermite cubic interpolant. For this choice of parameters (2.5) holds with $k=4$ and $C(\alpha,\beta)=1/384$.
2. If $\alpha=-1/8$, $\beta=-1$, then f is the Hermite quadratic interpolant, i.e. the quadratic C^1 spline interpolant with one knot at the midpoint of the initial interval. In this case (2.5) holds with $k=3$ and $C(\alpha,\beta)=1/96$, see [15].
3. The EQS-case $\alpha=\frac{\beta}{4(1-\beta)}$ with $\beta\in[-1,0)$ is a one parameter extension of the quadratic spline case. It was introduced and studied in [15], see also [17]. In this case (2.5) only holds with $k=2$ and $C(\alpha,\beta)\leq 1/48$ unless $\beta=-1$, but as we will see this scheme has important shape preserving properties.

2.2. Direct computation of the function or the derivative. We can reformulate (2.2),(2.3) so that only values of p are involved and similarly for f.

Proposition 2.1. *For $\alpha,\beta\in\mathbb{R}$, the function f and the derivative p of the HC^1-interpolant satisfy the following relations:*
For $n=1,2,\dots$

and $i = 0, 1, \ldots, 2^{n-1} - 1$,

$$\begin{bmatrix} p_{4i}^{n+1} \\ p_{4i+1}^{n+1} \\ p_{4i+2}^{n+1} \\ p_{4i+3}^{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \mu & 1+\beta/2 & -\nu \\ 0 & 1 & 0 \\ -\nu & 1+\beta/2 & \mu \end{bmatrix} \begin{bmatrix} p_{2i}^{n} \\ p_{2i+1}^{n} \\ p_{2i+2}^{n}, \end{bmatrix}. \tag{2.6}$$

For $n \geq 2$ *and* $i = 0, 1, \ldots 2^{n-2} - 1$

$$\begin{bmatrix} f_{8i}^{n+1} \\ f_{8i+1}^{n+1} \\ f_{8i+2}^{n+1} \\ f_{8i+3}^{n+1} \\ f_{8i+4}^{n+1} \\ f_{8i+5}^{n+1} \\ f_{8i+6}^{n+1} \\ f_{8i+7}^{n+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 1+\mu & 2(2-\mu) & \mu+\nu-1 & -2\nu & \nu \\ 0 & 4 & 0 & 0 & 0 \\ -\mu & 2(1+\mu) & 2-\mu-\nu & 2\nu & -\nu \\ 0 & 0 & 4 & 0 & 0 \\ -\nu & 2\nu & 2-\mu-\nu & 2(1+\mu) & -\mu \\ 0 & 0 & 0 & 4 & 0 \\ \nu & -2\nu & \mu+\nu-1 & 2(2-\mu) & 1+\mu \end{bmatrix} \begin{bmatrix} f_{4i}^{n} \\ f_{4i+1}^{n} \\ f_{4i+2}^{n} \\ f_{4i+3}^{n} \\ f_{4i+4}^{n}, \end{bmatrix} \tag{2.7}$$

where $\mu := -2\alpha(1-\beta)$ *and* $\nu = \mu + \beta/2$.

Proof. The result is clear for equations corresponding to even subscripts of p and f since the scheme is interpolating. Consider therefore the odd subscript equations. We will use the notation $\Delta p_k^n = p_{k+1}^n - p_k^n$, $\Delta f_k^n = f_{k+1}^n - f_k^n$ and $\Delta^2 f_k^n = \Delta(\Delta f_k^n) = f_{k+2}^n - 2f_{k+1}^n + f_k^n$.

Let us start by proving (2.6). Using (2.3) with $k = 2i$ and $k = 2i+1$

$$\begin{aligned} p_{4i+1}^{n+1} &= (1-\beta)\frac{\Delta f_{2i}^n}{h_n} + \frac{\beta}{2}\big(p_{2i+1}^n + p_{2i}^n\big) \\ p_{4i+3}^{n+1} &= (1-\beta)\frac{\Delta f_{2i+1}^n}{h_n} + \frac{\beta}{2}\big(p_{2i+2}^n + p_{2i+1}^n\big). \end{aligned} \tag{2.8}$$

From (2.2) we obtain

$$\begin{aligned} \frac{\Delta f_{2i}^n}{h_n} &= \frac{\Delta f_i^{n-1}}{h_{n-1}} + 2\alpha \Delta p_i^{n-1} \\ \frac{\Delta f_{2i+1}^n}{h_n} &= \frac{\Delta f_i^{n-1}}{h_{n-1}} - 2\alpha \Delta p_i^{n-1}, \end{aligned} \tag{2.9}$$

The f difference on the right can be eliminated by a reordering of (2.3) with $k = i$ and $n \to n-1$

$$(1-\beta)\frac{\Delta f_i^{n-1}}{h_{n-1}} = p_{2i+1}^n - \frac{\beta}{2}\big(p_{i+1}^{n-1} + p_i^{n-1}\big). \tag{2.10}$$

Combining (2.8)-(2.10), we find

$$\begin{aligned} p_{4i+1}^{n+1} &= p_{2i+1}^n + \frac{\beta}{2}\big(p_{2i+1}^n - p_{i+1}^{n-1}\big) - \mu \Delta p_i^{n-1} \\ p_{4i+3}^{n+1} &= p_{2i+1}^n + \frac{\beta}{2}\big(p_{2i+1}^n - p_i^{n-1}\big) + \mu \Delta p_i^{n-1} \end{aligned}$$

and we obtain (2.6).

In terms of differences (2.6) takes the form

$$\begin{aligned} \Delta p_{4i}^n &= (1-\mu)\Delta p_{2i}^{n-1} - \nu \Delta p_{2i+1}^{n-1} \\ \Delta p_{4i+1}^n &= \mu \Delta p_{2i}^{n-1} + \nu \Delta p_{2i+1}^{n-1} \\ \Delta p_{4i+2}^n &= \nu \Delta p_{2i}^{n-1} + \mu \Delta p_{2i+1}^{n-1} \\ \Delta p_{4i+3}^n &= -\nu \Delta p_{2i}^{n-1} + (1-\mu)\Delta p_{2i+1}^{n-1}. \end{aligned} \tag{2.11}$$

Notice that an equivalent formulation of (2.2) is

$$\Delta^2 f_{2k}^{n+1} = -2\alpha h_n \Delta p_k^n$$

and (2.11) can be written

$$\begin{aligned}
2\Delta^2 f_{8i}^{n+1} &= (1-\mu)\Delta^2 f_{4i}^n - \nu\Delta^2 f_{4i+2}^n \\
2\Delta^2 f_{8i+2}^{n+1} &= \mu\Delta^2 f_{4i}^n + \nu\Delta^2 f_{4i+2}^n \\
2\Delta^2 f_{8i+4}^{n+1} &= \nu\Delta^2 f_{4i}^n + \mu\Delta^2 f_{4i+2}^n \\
2\Delta^2 f_{8i+6}^{n+1} &= -\nu\Delta^2 f_{4i}^n + (1-\mu)\Delta^2 f_{4i+2}^n.
\end{aligned} \tag{2.12}$$

It remains to extract the values $f_{8i+j}^{n+1}, j = 1,3,5,7$ from the previous formulae to obtain (2.7). $\square$

From (2.6) it follows that the new p-values on level $n+1$ ($n \geq 1$) can be formed by an affine combination of three p values on the previous level n. This can especially be used to simplify the proofs of two results in [15] on monotonicity and convexity of the HC^1-interpolant.

For monotonicity the HC^1-algorithm is applied in [15, 3] to test data $(f_0, p_0, f_1, p_1) = (0, x, 1, y)$ computing the corresponding HC^1-interpolant f and its derivative p. For fixed (α, β) the authors determine the set of slopes (x, y) giving $p \geq 0$. Theorem 11 in [15] states that if $-1 < \beta < 0$ and $0 > \alpha \geq \beta/(4(1-\beta))$ then

$$M(\alpha,\beta) := \{(x,y) \in \mathbb{R}_+^2 : p \geq 0\} = \{(x,y) \in \mathbb{R}_+^2 : x + y \leq \gamma\} =: T(\gamma),$$

where $\gamma := \frac{2(\beta-1)}{\beta}$ and $\mathbb{R}_+^2 = \{(x,y) \in \mathbb{R} : x > 0, y > 0\}$. Note that any point in $\mathbb{R}_+^2$ belongs to $T(\gamma)$ for some $\beta < 0$. Thus we can obtain an increasing interpolant for any nonnegative initial slopes x, y by choosing β suitably close to zero. For arbitrary initial data (f_0, p_0, f_1, p_1) on $[a, b]$ one can use the change of variables $g(t) := \big(f(a + t(b-a)) - f_0)\big)/(f_1 - f_0)$ to show that the HC^1-interpolant f is increasing if and only if $\big(p_0/\Delta, p_1/\Delta\big) \in M(\alpha, \beta)$, where $\Delta := (f_1 - f_0)/(b-a)$.

The proof of Theorem 11 follows immediately from (2.6). Indeed for the assumed range of (α, β) the elements in the matrix in (2.6) are all nonnegative. Thus if p is nonnegative on level $n-1$ it is nonnegative on level n. Moreover, if $(x, y) \in T(\gamma)$ then $p((a+b)/2) = 1 - \beta + \beta(x+y)/2 \geq 1 - \beta + \frac{\beta}{2}\frac{2}{\beta}(\beta - 1) = 0$ and the theorem follows. In fact the theorem holds for $-2 \leq \beta < 0$ as long as we have C^1-convergence of the HC^1-interpolant. See the next subsection for convergence results.

For convexity the HC^1-algorithm is applied to the test data $(f_0, p_0, f_1, p_1) = (0, -x, 0, y)$ with $(x, y) \in \mathbb{R}_+^2$. For fixed (α, β) the set of slopes (x, y) giving an increasing p, is determined. Theorem 18 in [15] states that if $-1 \leq \beta < 0$ and $\gamma := (\beta - 2)/\beta$ then

$$C(\alpha,\beta) := \{(x,y) \in \mathbb{R}_+^2 : p \text{ is increasing}\} = \{(x,y) \in \mathbb{R}_+^2 : x/\gamma \leq y \leq x\gamma\} =: C^*(\gamma)$$

if and only if $\alpha = \beta/(4(1-\beta))$. Since any point in $\mathbb{R}_+^2$ belongs to $C^*(\gamma)$ for β sufficiently close to zero, this result implies that we can obtain a convex HC^1-interpolant in the EQS-case by using any nonnegative values (x, y) as initial data. For arbitrary initial data (f_0, p_0, f_1, p_1) on $[a, b]$ with $h = b - a$ one can for convexity use the change of variables $g(t) := f(a + th) - (1-t)f_0 - tf_1$ to show that the HC^1-interpolant f is convex if and only if $h * \big(p_1 - \Delta, \Delta - p_0\big) \in C(\alpha, \beta)$, where as before $\Delta := (f_1 - f_0)/h$.

To prove Theorem 18 we use (2.11). For the given value of α we have $\nu = 0$ and moreover $0 \leq \mu \leq 1$. Thus p is increasing on level n if it is increasing on level $n-1$. Since $-x = \beta(\gamma-1)x \leq \beta(y-x)/2 = p((a+b)/2) \leq \beta(y - \gamma y)/2 = y$, p is increasing on level 1 and the if part of the Theorem follows. The only if part is easy, see [15, p. 293].

In the EQS-case we only need two of the three p-values on the right of (2.6). Moreover the derivatives will be sampled from a piecewise linear curve.

Corollary 2.2. *In the EQS-case $\alpha = \frac{\beta}{4(1-\beta)}$ we have*

$$\begin{aligned} p^{n+1}_{4i+1} &= -\frac{\beta}{2}p^n_{2i} + (1+\frac{\beta}{2})p^n_{2i+1}, \\ p^{n+1}_{4i+3} &= (1+\frac{\beta}{2})p^n_{2i+1} - \frac{\beta}{2}p^n_{2i+2}. \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} 4f^{n+1}_{8i+1} =&(1-\frac{\beta}{2})f^n_{4i} + (4+\beta)f^n_{4i+1} - (1+\frac{\beta}{2})f^n_{4i+2} \\ 4f^{n+1}_{8i+3} =&\frac{\beta}{2}f^n_{4i} + (2-\beta)f^n_{4i+1} + (2+\frac{\beta}{2})f^n_{4i+2} \\ 4f^{n+1}_{8i+5} =&(2+\frac{\beta}{2})f^n_{4i+2} + (2-\beta)f^n_{4i+3} + \frac{\beta}{2}f^n_{4i+4} \\ 4f^{n+1}_{8i+7} =&-(1+\frac{\beta}{2})f^n_{4i+2} + (4+\beta)f^n_{4i+3} + (1-\frac{\beta}{2})f^n_{4i+4} \end{aligned} \tag{2.14}$$

If in addition $\beta \in (-2, 0)$ then there exist

$$a = \tau^n_0 < \tau^n_1 < \cdots < \tau^n_{2^n} = b, \tag{2.15}$$

with $\tau^n_{2^{n-1}} = \frac{a+b}{2}$ for $n \geq 1$. such that

$$p^n_i = L(\tau^n_i), \quad i = 0, 1, \ldots, 2^n, \quad n = 0, 1, \ldots, \tag{2.16}$$

where L is the piecewise linear curve connecting the three points $(a, p(a))$, $(\frac{a+b}{2}, p(\frac{a+b}{2}))$, $(b, p(b))$.

Proof. If $\alpha = \frac{\beta}{4(1-\beta)}$ then $\mu = -\beta/2$ and (2.13) follows from (2.6). Similarly, we obtain (2.14). We claim that (2.16) holds with

$$\begin{aligned} \tau^{n+1}_{4i} &= \tau^n_{2i}, \quad \tau^{n+1}_{4i+1} = -\frac{\beta}{2}\tau^n_{2i} + (1+\frac{\beta}{2})\tau^n_{2i+1}, \\ \tau^{n+1}_{4i+2} &= \tau^n_{2i+1}, \quad \tau^{n+1}_{4i+3} = -\frac{\beta}{2}\tau^n_{2i+2} + (1+\frac{\beta}{2})\tau^n_{2i+1}. \end{aligned} \tag{2.17}$$

Since $p^n_0 = p(a)$ and $p^n_{2^n} = p(b)$, we have $\tau^n_0 = a$ and $\tau^n_{2^n} = b$ for all $n \geq 0$. Moreover, since $p^n_{2^{n-1}} = p(\frac{a+b}{2})$, we see that $\tau^n_{2^{n-1}} = \frac{a+b}{2}$ for all $n \geq 1$. Thus (2.15) will follow from (2.17) since the latter involves convex combinations for $\beta \in (-2, 0)$. (2.17) follows from (2.13) by induction. Suppose (2.16) holds for some n. Since L is linear on the actual segment we obtain

$$p^{n+1}_{4i+1} = -\frac{\beta}{2}L(\tau^n_{2i}) + (1+\frac{\beta}{2})L(\tau^n_{2i+1}) = L(\tau^{n+1}_{4i+1}),$$

where τ^{n+1}_{4i+1} is given by (2.17). The proof of the other τ-relation is similar. □

2.3. C^1-convergence. To study convergence we observe that it is enough to consider the interval $[0, 1]$. Indeed, if $I := [a, b]$ and $h := b - a$, defining the initial data $g(u) = f(a + hu), g'(u) = hf'(a + hu)$, for $u \in \{0, 1\}$, the construction of f on $[a, b]$ or g on $[0, 1]$ by (2.1) are equivalent and at step n, $g(u) = f(a + uh)$ and $g'(u) = hf'(a + hu)$ for $u \in \{0, 1/2^n, \ldots, \ell/2^n, \ldots, 1\}$.

In [14] it was shown that if there exist positive constants c, ρ with $\rho < 1$ such that for each integer $n \geq 0$ we have $|\Delta p^n_i| \leq c\rho^n$ for $i = 0, 1, \ldots, 2^n - 1$, where

$$\Delta p^n_i := p\big(\frac{i+1}{2^n}\big) - p\big(\frac{i}{2^n}\big), \quad i = 0, 1, \ldots, 2^n - 1, \tag{2.18}$$

then p has a unique continuous extension to I. Moreover, there is a positive constant c_1 such that for all $(x,y) \in [0,1]^2$

$$|p(x) - p(y)| \leq c_1 |x-y|^{-\log_2 \rho},$$

i.e. p is Hölder continuous with exponent $-\log_2 \rho$.

Suppose p is continuous and $\lim\limits_{n\to\infty} \max\limits_{0\leq i<2^n-1} |\Delta(f,p)_i^n| = 0$, where

$$\Delta(f,p)_i^n := 2^n \Delta f_i^n - \sigma p_i^n, \quad \sigma p_i^n := \frac{1}{2}\big(p\big(\frac{i+1}{2^n}\big) + p\big(\frac{i}{2^n}\big)\big), \tag{2.19}$$

and $\Delta f_i^n = f\big(\frac{i+1}{2^n}\big) - f\big(\frac{i}{2^n}\big)$. Then ([14]) f has a unique continuous extension to $I := [0,1]$. Moreover $f \in C^1([0,1])$ with $f' = p$. From this discussion we have the following lemma.

Lemma 2.3. *Let $U_i^n := [\Delta p_i^n, \Delta(f,p)_i^n]^T$ for $i = 0,1,\ldots,2^n-1$ and $n = 0,1,2,\ldots$. If we can find a vector norm $\|\cdot\|$ on $\mathbb{R}^2$ and positive constants c, ρ with $\rho < 1$ such that*

$$\|U_i^n\| \leq c\rho^n, \; i = 0,1,\ldots,2^n-1 \text{ and } n = 0,1,\ldots$$

then the HC^1-algorithm is C^1-convergent and $f' = p$ is Hölder continuous with exponent $-\log_2 \rho$.

We can now show

Proposition 2.4. *Algorithm HC^1 is C^1-convergent for $(\alpha,\beta) \in [-1/8, 0) \times [-2, 1)$.*

Proof. An immediate evaluation gives

$$U_{2i}^{n+1} = \Lambda_1 U_i^n \text{ and } U_{2i+1}^{n+1} = \Lambda_{-1} U_i^n \text{ for } i = 0,1,\ldots,2^n, n = 0,1,\ldots,$$

where

$$\Lambda_\epsilon = \begin{bmatrix} \dfrac{1}{2} & \epsilon(1-\beta) \\ \epsilon\dfrac{8\alpha+1}{4} & \dfrac{1+\beta}{2} \end{bmatrix}, \quad \varepsilon = \pm 1. \tag{2.20}$$

Note that the off-diagonal elements of Λ_ε have the same sign for $\alpha \geq -1/8$ and $\beta \leq 1$. We define a vector norm by $\|v\| := \|P^{-1}v\|_2$, where $\|\cdot\|_2$ is the usual Euclidian vector norm and $P := \begin{bmatrix} 2\sqrt{1-\beta} & 0 \\ 0 & \sqrt{8\alpha+1} \end{bmatrix}$. Then $P^{-1}\Lambda_\varepsilon P$ is symmetric and the corresponding matrix operator norm is given by $\|\Lambda_\varepsilon\| := \|P^{-1}\Lambda_\varepsilon P\|_2$, where $\|A\|_2 := \sqrt{\rho(A^TA)}$ is the spectral norm of a matrix A. The eigenvalues of Λ_ε or of $P^{-1}\Lambda_\varepsilon P$ are

$$\lambda_1 = \frac{1}{4}\big(2+\beta+\sqrt{(2-\beta)^2+32\alpha(1-\beta)}\big), \quad \lambda_2 = \frac{1}{4}\big(2+\beta-\sqrt{(2-\beta)^2+32\alpha(1-\beta)}\big)$$

Since $P^{-1}\Lambda_\varepsilon P$ is symmetric the eigenvalues are real with $\lambda_2 < \lambda_1$. Now for $\beta \in [-2,1)$ and $\alpha \in [-1/8, 0)$ we find $\lambda_1 < (2+\beta+\sqrt{(2-\beta)^2})/4 = 1$ and $\lambda_2 > (2+\beta-\sqrt{(2-\beta)^2})/4 = \beta/2 \geq -1$. Thus $\rho := \|\Lambda_\varepsilon\| = \max\{|\lambda_1|, |\lambda_2|\} < 1$ for $\varepsilon = \pm 1$ and we have shown that $\max\{\|U_{2i}^{n+1}\|, \|U_{2i+1}^{n+1}\|\} \leq \rho\|U_i^n\|$ so that $\|U_i^n\| \leq \rho^n \|U_0^0\|$ for $i = 0,1,\ldots,2^n-1$ and $n = 0,1,2,\cdots$. The C^1-convergence for $(\alpha,\beta) \in [-1/8, 0) \times [-2, 1)$ now follows from Lemma 2.3 ☐

By Proposition 2.4, the HC^1-algorithm converges for $\beta \in [-1, 0)$ if $\alpha = \frac{\beta}{4(1-\beta)}$. We can now extend this result.

Proposition 2.5. *If $\alpha = \frac{\beta}{4(1-\beta)}$ then the HC^1-algorithm is C^1-convergent for $\beta \in (-2, 0)$.*

Proof. For $\varepsilon = \pm 1$ the matrices Λ_ε in (2.20) take the form : $\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon(1-\beta) \\ \varepsilon\frac{\beta+1}{4(1-\beta)} & \frac{1+\beta}{2} \end{pmatrix}$.

Now, for any positive real number θ, we define the norm $\|\cdot\|_\theta$ on $\mathbb{R}^2$ by $\|(x,y)\|_\theta = |x| + \theta|y|$. It is easy to prove that for any matrix $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$, the corresponding matrix operator norm is given by $\|M\|_\theta := \max(|m_{11}| + \theta|m_{21}|, |m_{12}|/\theta + |m_{22}|)$. Choosing $\theta = 2(1-\beta)$ we find $\|\Lambda_1\|_\theta = \|\Lambda_{-1}\|_\theta = 1/2(1+|1+\beta|)$, which is stricly less than one for $-2 < \beta < 0$. Lemma 2.3 now gives the convergence.
□

We define the convergence region C by

$$C := \big\{(\alpha,\beta) : \text{ the scheme } HC^1 \text{ is } C^1 \text{ - convergent}\big\}. \tag{2.21}$$

We have shown that $[-1/8, 0) \times [-2, 1) \subset C$ and also that $\{(\frac{\beta}{4(1-\beta)}, \beta) : -2 < \beta < 0\} \subset C$.

The function $f' = p$ is Hölder continuous with exponent $-\log_2 \rho$. In the case where $\alpha = \frac{\beta}{4(1-\beta)}$ we have $\|\Lambda_1\|_\theta = \|\Lambda_{-1}\|_\theta = \rho = \rho(\beta) = 1/2(1+|1+\beta|)$ which is piecewise linear with a minimum for $\beta = -1$ and we obtain the best regularity of the interpolant for $\beta = -1$ when f is a quadratic spline.

To illustrate the smoothness properties of a HC^1-interpolant we show the Hermite basis with $\beta = -3/5$ and $\alpha = \frac{\beta}{4(1-\beta)} = -3/32$ in Figure 2.1. The spectral radius of the matrices Λ_ε is $7/10$ and hence the derivatives of the Hermite basis functions are Hölder continuous with exponent $\rho = -\log_2(7/10) \approx 0.5146$.

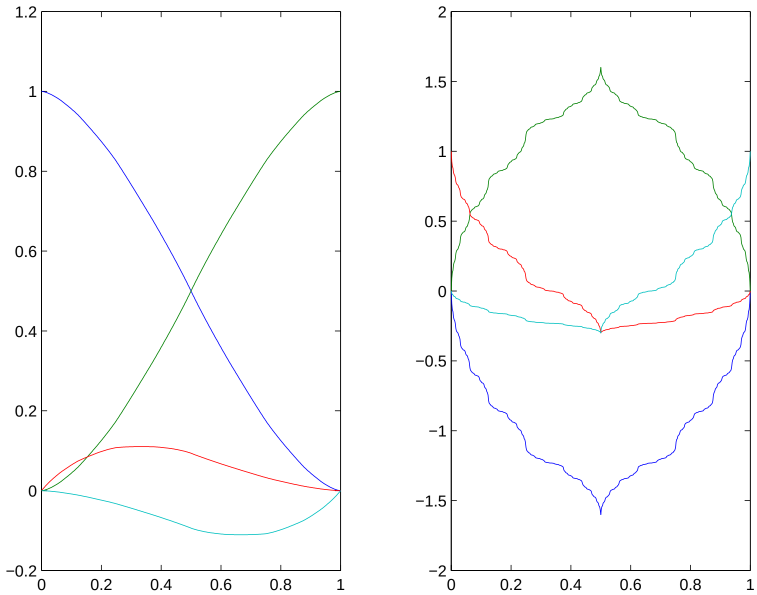

FIG. 2.1. *Hermite basis and derivatives, corresponding to* $\alpha = -3/32$ *and* $\beta = -3/5$.

Remark: The data $f(a), p(a), f(b), p(b)$ can either have real values or vector values in $\mathbb{R}^s, s \geq 2$. In this second case, we look for vector continuous functions f and p with $f' = p$ from $I = [a,b]$ to $\mathbb{R}^s$. The C^1-convergence is guaranteed for all (α,β) in the convergence region C since it suffices to study the convergence independently for each component of f and p.

3. Control Polygons and Subdivision Algorithm.

3.1. Control Coefficients and Control Polygons. Suppose we apply the subdivision scheme HC^1 to some real valued data $f(a)$, $p(a)$, $f(b)$, $p(b)$. In order to obtain a geometric formulation of the scheme we define *control coefficients* relative to the interval $[a,b]$ by

$$a_0 = f(a), \quad a_1 = f(a) + \frac{h}{\lambda}p(a), \quad a_2 = f(b) - \frac{h}{\lambda}p(b), \quad a_3 = f(b), \tag{3.1}$$

where $h := b-a$ and $\lambda \geq 2$ is a real number to be chosen. We define the *control points* (A_0, A_1, A_2, A_3) on $[a,b]$ by

$$A_0 = (a, a_0), \quad A_1 = (a + \frac{h}{\lambda}, a_1), \quad A_2 = (b - \frac{h}{\lambda}, a_2), \quad A_3 = (b, a_3), \tag{3.2}$$

and the *control polygon* $\{A_0, A_1, A_2, A_3\}$ on $[a,b]$ by connecting the four control points by straight line segments. If f is the HC^1-interpolant then the parametric curve $(x, f(x))$ with $x \in [a,b]$ passes through A_0 with tangent directions $A_1 - A_0$ and A_3 with tangent direction $A_3 - A_2$. See Figure 3.1.

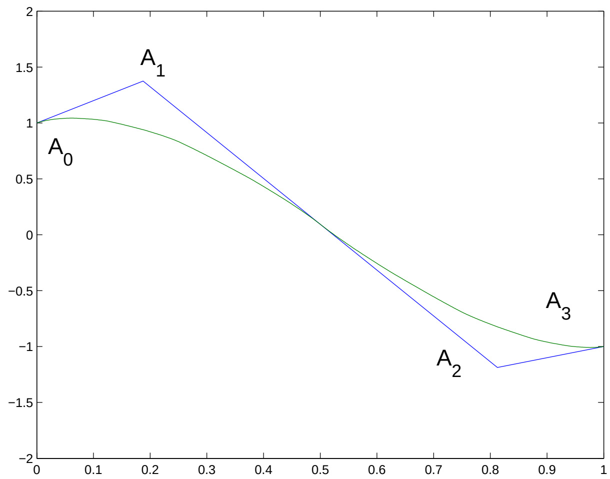

FIG. 3.1. *A HC^1-interpolant and its control polygon, $\beta = -3/5$, $\alpha = -3/32, \lambda = 16/3$.*

We can also apply the subdivision scheme HC^1 to vector valued data f_0, p_0, f_1, p_1 in $\mathbb{R}^s$ for some $s \geq 2$. We pick an interval $[a,b]$ and use the HC^1-algorithm on each component of f and p. To obtain a geometric formulation of this process we define control coefficients relative to $[a,b]$ by (3.1) and we define the control points to be the same as the control coefficients. The computed curve interpolates the first and last control coefficient and its tangent direction at a_0 is $a_1 - a_0$, and at a_3 the tangent direction is $a_3 - a_2$.

Note that if 4 points a_0, a_1, a_2, a_3 in $\mathbb{R}^s$ for $s \geq 1$ are given we can think of these as control coefficients of a HC^1-interpolant on some finite interval $[a,b]$ and apply the HC^1 algorithm to the data given by

$$f(a) := a_0, \; p(a) := \frac{\lambda}{h}\big(a_1 - a_0\big), \; f(b) := a_3, \; p(b) := \frac{\lambda}{h}\big(a_3 - a_2\big), \tag{3.3}$$

where $h := b-a$. We now derive a parameter independent formulation of this scheme. In particular suppose (a_0, a_1, a_2, a_3) are points in $\mathbb{R}^s$ for some $s \geq 1$ which are distinct if $s \geq 2$ and let $[a,b]$ be any finite interval.

Using (3.1) and (3.3) we can compute new control coefficients $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)$ for the interval I_1 and new control coefficients $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)$ for I_2, and then join them into control coefficients $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)$ on $[a,b]$. In the following geometric formulation of the subdivision scheme we do this computation directly without picking an underlying interval $[a,b]$. The proposition is a generalization of [17, Theorem 10]:

Proposition 3.1. *Suppose $a_i \in \mathbb{R}^s$ for $i = 0,1,2,3$ and some $s \geq 1$. After one subdivision of the control coefficients (a_0,a_1,a_2,a_3) we obtain new control coefficients $(\bar{a}_0,\bar{a}_1,\bar{a}_2,\bar{a}_3,\bar{a}_4,\bar{a}_5,\bar{a}_6)$ given by*

$$\begin{bmatrix}\bar{a}_0\\ \bar{a}_1\\ \bar{a}_2\\ \bar{a}_3\\ \bar{a}_4\\ \bar{a}_5\\ \bar{a}_6\end{bmatrix} = \mathbf{S}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix} := \frac{1}{4}\begin{bmatrix}4 & 0 & 0 & 0\\ 2 & 2 & 0 & 0\\ \gamma & v-\beta & v+\beta & \delta\\ 2-v & v & v & 2-v\\ \delta & v+\beta & v-\beta & \gamma\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 4\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}, \tag{3.4}$$

where

$$\begin{aligned} v &= -4\alpha\lambda\\ \gamma &= 2 - v + (2+\beta(\lambda-2))/\lambda\\ \delta &= 2 - v - (2+\beta(\lambda-2))/\lambda. \end{aligned} \tag{3.5}$$

Moreover,

$$\bar{a}_3 = \frac{1}{2}(\bar{a}_2 + \bar{a}_4). \tag{3.6}$$

Proof. Pick any interval $[a,b]$ and let $h := b-a$. By (3.1)

$$\begin{aligned} &\bar{a}_0 = f(a),\ \bar{a}_1 = f(a) + \frac{h}{2\lambda}p(a),\ \bar{a}_2 = f(\frac{a+b}{2}) - \frac{h}{2\lambda}p(\frac{a+b}{2}),\ \bar{a}_3 = f(\frac{a+b}{2}),\\ &\bar{a}_6 = f(b),\ \bar{a}_5 = f(b) - \frac{h}{2\lambda}p(b),\ \bar{a}_4 = f(\frac{a+b}{2}) + \frac{h}{2\lambda}p(\frac{a+b}{2}). \end{aligned}$$

From (2.1) and (3.3) we obtain on an interval $[a,b]$ the inverse relations

$$\begin{aligned} f(a) = a_0, \quad p(a) &= \frac{\lambda}{h}(a_1 - a_0)\\ f(b) = a_3, \quad p(b) &= \frac{\lambda}{h}(a_3 - a_2)\\ f(\frac{a+b}{2}) &= \frac{a_0+a_3}{2} - \frac{v}{4}(a_3 - a_2 - a_1 + a_0)\\ \frac{h}{2\lambda}p(\frac{a+b}{2}) &= \frac{1-\beta}{2\lambda}(a_3 - a_0) + \frac{\beta}{4}(a_3 - a_2 + a_1 - a_0)\\ &= \frac{2+\beta(\lambda-2)}{\lambda}(a_3 - a_0) + \frac{\beta}{4}(a_1 - a_2). \end{aligned} \tag{3.7}$$

But then we see that $(\bar{a}_0,\bar{a}_1,\bar{a}_2,\bar{a}_3,\bar{a}_4,\bar{a}_5,\bar{a}_6)^T = \mathbf{S}(a_0,\ldots,a_3)^T$, where $\mathbf{S}$ is the matrix in equation (3.4). Since the sum of rows three and five in the matrix $\mathbf{S}$ equals twice row four the relation (3.6) follows. ∎

For $s \geq 2$ the control coefficients and control points are the same and the proposition also gives rules for subdividing the control polygon. The following corollary holds in general.

Corollary 3.2. *Suppose $(a_0,a_1,a_2,a_3) \in \mathbb{R}^s$ for some $s \geq 1$. After one subdivision of the corresponding control polygon $\{A_0,A_1,A_2,A_3\}$ we obtain a new control polygon $\{\bar{A}_0,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4,\bar{A}_5,\bar{A}_6\}$ given by*

$$\begin{bmatrix}\bar{A}_0 & \bar{A}_1 & \bar{A}_2 & \bar{A}_3 & \bar{A}_4 & \bar{A}_5 & \bar{A}_6\end{bmatrix}^T = \mathbf{S}\begin{bmatrix}A_0 & A_1 & A_2 & A_3\end{bmatrix}^T \tag{3.8}$$

where $\mathbf{S}$ *is given by* (3.4)*. Moreover*

$$\bar{A}_3 = \frac{1}{2}(\bar{A}_2 + \bar{A}_4), \tag{3.9}$$

which means that these control points always lie on a straight line.

Proof. This has already been shown for $s \geq 2$ and for the control coefficients for $s = 1$. For the control point abscissas we obtain the relation $(a, a+h/(2\lambda), \bar{a}-h/(2\lambda), \bar{a}, \bar{a}+h/(2\lambda), b-h/(2\lambda), d)^T = \mathbf{S}(a, a+h/\lambda, b-h/\lambda, d)^T$, where $\bar{a} = (a+b)/2$, since the scheme HC^1 reproduces linear functions. Thus (3.8) and (3.9) also holds for $s = 1$. $\square$

3.2. A Stationary Subdivision Algorithm. By applying (3.4), we can reformulate the Hermite subdivision scheme HC^1 as a stationary subdivision scheme working on points in $\mathbb{R}^s$.

Starting with 4 points a_0, a_1, a_2, a_3 in $\mathbb{R}^s$, $s \geq 1$, (α, β) in the convergence region C, and $\lambda \geq 2$, we define Algorithm SC^1 as follows.

At step $n = 0$, we set $a_0^0 = a_0, a_1^0 = a_1, a_2^0 = a_2, a_3^0 = a_3$.

At step $n+1, n \geq 0$, we define

$$\begin{bmatrix} a_{6i}^{n+1} \\ a_{6i+1}^{n+1} \\ a_{6i+2}^{n+1} \\ a_{6i+3}^{n+1} \\ a_{6i+4}^{n+1} \\ a_{6i+5}^{n+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \gamma & v-\beta & v+\beta & \delta \\ 2-v & v & v & 2-v \\ \delta & v+\beta & v-\beta & \gamma \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} a_{3i}^n \\ a_{3i+1}^n \\ a_{3i+2}^n \\ a_{3i+3}^n \end{bmatrix}, \quad i = 0, 1, \ldots 2^n - 1 \tag{3.10}$$

and $a_{3.2^{n+1}}^{n+1} = a_{3.2^n}^n$. Here v, γ, δ are given by (3.5). The matrix $(s_{\ell,k})_{\ell=0,\ldots,5,k=0,\ldots,3}$ in (3.10) is formed from the first 6 rows of $\mathbf{S}$ given by (3.4).

LEMMA 3.3. *For all* $n \geq 1$ *and for all* $i = 1, \ldots, 2^n - 1$*, we have*

$$\begin{aligned} a_{6i}^{n+1} &= a_{3i}^n, \; i = 1, \ldots, 2^{n-1} \\ a_{6i+1}^{n+1} - a_{6i}^{n+1} &= \frac{1}{2}\big(a_{3i+1}^n - a_{3i}^n\big), \; i = 1, \ldots, 2^{n-1} - 1 \\ a_{3i+1}^n + a_{3i-1}^n &= 2a_{3i}^n, \; i = 1, \ldots, 2^n - 1 \end{aligned} \tag{3.11}$$

Proof. The first two equations follow immediately from (3.10). As in the proof of (3.6) it is clear that

$$a_{6i+2}^{n+1} + a_{6i+4}^{n+1} = 2a_{6i+3}^{n+1}, \quad i = 0, \ldots, 2^n - 1, \quad n = 0, 1, \ldots,$$

and in particular $a_2^1 + a_4^1 = 2a_3^1$. By (3.11) and induction on n

$$a_{6i+1}^{n+1} + a_{6i-1}^{n+1} = \frac{1}{2}(a_{3i}^n + a_{3i+1}^n) + \frac{1}{2}(a_{3i-1}^n + a_{3i}^n) = 2a_{3i}^n = 2a_{6i}^{n+1}.$$

$\square$

If we define a_i^0 for $i < 0$ and $i > 3$ in any way, the subdivision scheme can be written $a_\ell^{n+1} = \sum_{k \in \mathbb{Z}} \sigma_{\ell,k} a_k^n, \ell \in \mathbb{Z}$ where $\sigma_{6i+\ell,3i+k} = s_{\ell,k}$ for $i \in \mathbb{Z}, \ell = 0, \ldots, 5, k = 0, \ldots, 3$ and $\sigma_{i,j} = 0$ otherwise. With the definitions recalled in [3], the scheme is *local* since $\sigma_{\ell,k} = 0$ for $|\ell - 2k| > 4$. Since $\sum_{k \in \mathbb{Z}} \sigma_{\ell,k} = 1$, it is *affine* but it is not interpolating in a classical sense since we generally have $a_{6i+2}^{n+1} \neq a_{3i+1}^n$.

3.3. Convergence of SC^1. The convergence of the subdivision schemes are usually established by studying the difference sequence. Alternatively convergence follows since SC^1 was derived from HC^1. Here are the details.

THEOREM 3.4. *Let $s \geq 1$ and a_0, a_1, a_2, a_3 be 4 points in $\mathbb{R}^s$. Suppose $\lambda \geq 2$ and that (α, β) is in the convergence region C given by (2.21). We build the sequence of points $\{ a_i^n\}_{n\in\mathbb{N}, i=0,\dots,3.2^n}$ by (3.10). Choose any interval $I := [a, b]$ with $h := b - a > 0$ and define $t_i^n := a + ih_n$, where $h_n := h2^{-n}$ for $n \in \mathbb{N}$ and $i = 0, \dots, 2^n$. Then, there exists a C^1 function $f : I \to \mathbb{R}^s$ such that for all $n \in \mathbb{N}$:*

$$a_{3i}^n = f(t_i^n), \quad i = 0, \dots, 2^n,$$
$$a_{3i+1}^n - a_{3i}^n = \frac{h_n}{\lambda} f'(t_i^n), \quad i = 0, \dots, 2^n - 1,$$
$$a_{3i}^n - a_{3i-1}^n = \frac{h_n}{\lambda} f'(t_i^n), \quad i = 1, \dots, 2^n.$$

For $s \geq 2$, let $A_i^n = a_i^n$, $i = 0, 1, \dots, 3 \times 2^n$ and for $s = 1$, let $A_{3i}^n = (t_i^n, a_{3i}^n)$,$A_{3i+1}^n = (t_i^n + \frac{h_n}{\lambda}, a_{3i+1}^n)$, $A_{3i+2}^n = (t_{i+1}^n - \frac{h_n}{\lambda}, a_{3i+2}^n)$, $i = 0, 1, \dots, 2^n - 1$, and $A_{3\times 2^n}^n = (b, a_{3\times 2^n}^n)$.

Then the sequence of polygons $\{A_0^n, \dots, A_{3\times 2^n}^n\}$ converges to the curve $\{f(t), t \in I\}$.

Proof. We will show that the scheme SC^1 generates sequences $\{f^n\}$ and $\{p^n\}$ of piecewise linear vector functions which interpolate values and derivatives at the points of $\mathcal{P}_n = \{t_0^n, \dots, t_{2^n}^n\}$.

We define f^n and p^n to be linear on $[t_i^n, t_{i+1}^n], i = 0, \dots, 2^n - 1$, and to interpolate the following values

$$\begin{aligned} f^n(t_i^n) &= a_{3i}^n, \quad p^n(t_i^n) = \frac{\lambda}{h_n}(a_{3i+1}^n - a_{3i}^n), \quad i = 0, \dots, 2^n - 1, \\ f^n(b) &= a_{3.2^n}^n, \quad p^n(b) = \frac{\lambda}{h_n}(a_{3\times 2^n}^n - a_{3\times 2^n-1}^n). \end{aligned} \tag{3.12}$$

Since $t_i^n = t_{2i}^{n+1}$ we find from (3.11) and (3.12)

$$f^{n+1}(t_i^n) = f^n(t_i^n), \quad p^{n+1}(t_i^n) = p^n(t_i^n), \; i = 0, \dots, 2^n. \tag{3.13}$$

Below we prove that, for $i = 0, \dots, 2^n - 1$,

$$f^{n+1}(t_{2i+1}^{n+1}) = \frac{f^n(t_{i+1}^n) + f^n(t_i^n)}{2} + \alpha h_n(p^n(t_{i+1}^n) - p^n(t_i^n)), \tag{3.14}$$

$$p^{n+1}(t_{2i+1}^{n+1}) = (1 - \beta)\frac{f^n(t_{i+1}^n) - f^n(t_i^n)}{h_n} + \beta\frac{p^n(t_{i+1}^n) + p^n(t_i^n)}{2}. \tag{3.15}$$

Comparing (3.13), (3.14) and (3.15) with (2.2)-(2.3) we conclude that $f^n = f$ and $p^n = p$ on $\mathcal{P}_n$ where f and p are the functions built on $\cup\mathcal{P}_n$ by HC^1 defined by (2.2)-(2.4) from the initial data $f(a) = a_0$, $p(a) = \frac{\lambda}{h}(a_1 - a_0)$, $f(b) = a_3$ and $p(b) = \frac{\lambda}{h}(a_3 - a_2)$, and then extended to $[a, b]$. So that, if $(\alpha, \beta) \in C$, then the sequences f^n and p^n defined from SC^1 by (3.12) converge uniformly to continuous vector functions f and p defined on $[a, b]$. Moreover $f \in C^1([a, b])$ and $f' = p$.

Now since f' is bounded and $a_{3i+1}^n - a_{3i}^n = \frac{h}{\lambda 2^n} f'(t_i^n), i = 0, \dots, 2^n - 1$, we deduce that $a_{3i+1}^n - a_{3i}^n$ tends uniformly to 0. We conclude that the sequence of polygons $\{A_0, \dots, A_{3\times 2^n}\}$ tends to the curve $\{f(t), t \in I\}$ since $a_{3i}^n = f(t_i^n)$ for $i = 0, \dots, 2^n$.

It remains to prove (3.14) and (3.15). Since $\alpha = -v/4\lambda$, for $i = 0, \dots, 2^n - 1$ and using (3.12) and (3.10),

$$\begin{aligned} &\frac{1}{2}(f^n(t_{i+1}^n) + f^n(t_i^n)) + \alpha h_n(p^n(t_{i+1}^n) - p^n(t_i^n)) \\ &= \frac{1}{2}(a_{3i+3}^n + a_{3i}^n) - \frac{v}{4}(a_{3i+3}^n - a_{3i+2}^n - a_{3i+1}^n + a_{3i}^n) \\ &= a_{6i+3}^{n+1} = f^{n+1}(t_{2i+1}^{n+1}) \end{aligned}$$

so that (3.14) is proved.

Similarly, for (3.15), let $i \in \{0, \dots, 2^n - 1\}$. With the definitions of γ and δ in (3.5) we find

$$\begin{aligned}
&\frac{1-\beta}{h_n}(f^n(t^n_{i+1}) - f^n(t^n_i)) + \frac{\beta}{2}(p^n(t^n_{i+1}) + p^n(t^n_i)) \\
&\quad = \frac{1-\beta}{h_n}(a^n_{3i+3} - a^n_{3i}) + \frac{\beta\lambda}{2h_n}(a^n_{3i+3} - a^n_{3i+2} + a^n_{3i+1} - a^n_{3i}) \\
&\quad = \frac{\lambda}{h_{n+1}}\big(- (\frac{1-\beta}{2\lambda} + \frac{\beta}{4})a^n_{3i} + \frac{\beta}{4}a^n_{3i+1} - \frac{\beta}{4}a^n_{3i+2} + (\frac{1-\beta}{2\lambda} + \frac{\beta}{4})a^n_{3i+3}\big) \\
&\quad = \frac{1}{4}\frac{\lambda}{h_{n+1}}\big((\delta - 2 + v)a^n_{3i} + \beta a^n_{3i+1} - \beta a^n_{3i+2} + (\gamma - 2 + v)a^n_{3i+3}\big) \\
&\quad = \frac{\lambda}{h_{n+1}}\big(a^{n+1}_{6i+4} - a^{n+1}_{6i+3}\big) = p^{n+1}(t^{n+1}_{2i+1}).
\end{aligned}$$

□

4. Total positivity and consequences.

4.1. Corner Cutting and Total Positivity of the Subdivision Matrix. Consider now the subdivision process in the EQS-case when $\alpha = \frac{\beta}{4(1-\beta)}$ with $\beta \in (-2, 0)$. Since $v = -4\alpha\lambda = \frac{\beta}{\beta-1}\lambda$, or $\lambda = \frac{\beta-1}{\beta}v$ we find from (3.5)

$$\gamma = 2 - v + \frac{2 + \beta\lambda - 2\beta}{\lambda} = 2 - v + \frac{2 + (\beta-1)v - 2\beta}{(\beta-1)v}\beta = \frac{(2-v)(v-\beta)}{v},$$

and similarly

$$\delta = \frac{(2-v)(v+\beta)}{v}.$$

Thus the subdivision matrix (3.4) can be written

$$\mathbf{S} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \frac{(2-v)(v-\beta)}{v} & v-\beta & v+\beta & \frac{(2-v)(v+\beta)}{v} \\ 2-v & v & v & 2-v \\ \frac{(2-v)(v+\beta)}{v} & v+\beta & v-\beta & \frac{(2-v)(v-\beta)}{v} \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}. \tag{4.1}$$

In this case, as soon as $1 \le v \le 2$ and $v \ge -\beta$, we can compute the subdivided control points

$$(\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6)^T = \mathbf{S}(A_0, A_1, A_2, A_3)^T$$

by successive convex combinations starting with the polygon defined by (A_0, A_1, A_2, A_3). With 2 intermediate quantities B and C we have

$$\begin{aligned}
\bar{A}_0 &= A_0, \quad \bar{A}_1 = \frac{1}{2}A_0 + \frac{1}{2}A_1, \quad \bar{A}_5 = \frac{1}{2}A_2 + \frac{1}{2}A_3, \quad \bar{A}_6 = A_3,\\
B &= (1-\frac{v}{2})A_0 + \frac{v}{2}A_1\\
C &= (1-\frac{v}{2})A_3 + \frac{v}{2}A_2\\
\bar{A}_2 &= \frac{v-\beta}{2v}B + \frac{v+\beta}{2v}C\\
\bar{A}_4 &= \frac{v+\beta}{2v}B + \frac{v-\beta}{2v}C\\
\bar{A}_3 &= \frac{1}{2}\bar{A}_2 + \frac{1}{2}\bar{A}_4.
\end{aligned} \tag{4.2}$$

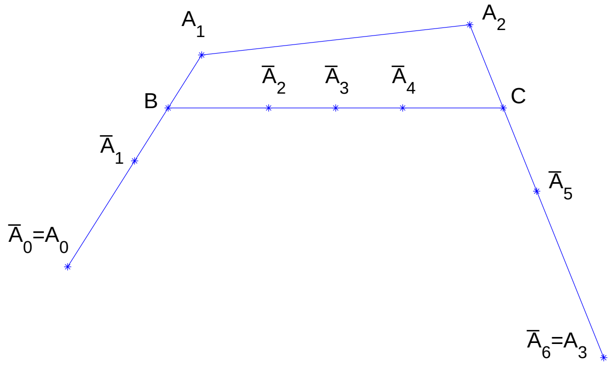

FIG. 4.1. *Corner Cutting with $\alpha = -3/32$, $\beta = -3/5$ and $v = 1.5$.*

For $v = 2$ we obtain $B = A_1$ and $C = A_2$. The value of λ corresponding to $v = 2$ was considered in [17, Theorem 10], where formulae similar to (4.2) were given.

The equations (4.2) can be formulated as a corner cutting scheme in the following way. We start with the polygon $\{A_0, A_1, A_2, A_3\}$ and then either cut one of the previous corners or break an edge in a sequence of convex combinations.

1. $B = (1-\frac{v}{2})A_0 + \frac{v}{2}A_1$ (replace A_1 by B to obtain $\{A_0, B, A_2, A_3\}$)
2. $C = (1-\frac{v}{2})A_3 + \frac{v}{2}A_2$ (replace A_2 by C to obtain $\{A_0, B, C, A_3\}$)
3. $\bar{A}_1 = (1-\frac{1}{v})A_0 + \frac{1}{v}B$ (break $[A_0, B]$ to obtain $\{A_0, \bar{A}_1, B, C, A_3\}$)
4. $\bar{A}_5 = \frac{1}{v}C + (1-\frac{1}{v})A_3$ (break $[C, A_3]$ to obtain $\{A_0, \bar{A}_1, B, C, \bar{A}_5, A_3\}$)
5. $\bar{A}_2 = \frac{v-\beta}{2v}B + \frac{v+\beta}{2v}C$ (replace B by $\bar{A}_2$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, C, \bar{A}_5, A_3\}$)
6. $\bar{A}_4 = \frac{v+\beta}{v-\beta}\bar{A}_2 - \frac{2\beta}{v-\beta}C$ (replace C by $\bar{A}_4$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, \bar{A}_4, \bar{A}_5, A_3\}$)
7. $\bar{A}_3 = (\bar{A}_2 + \bar{A}_4)/2$ (break $[\bar{A}_2, \bar{A}_4]$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, A_3\}$)

Since $\bar{A}_0 = A_0$ and $\bar{A}_6 = A_3$ we have obtained the subdivided polygon $\{\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6\}$ by carrying out a sequence of simple corner cuts (see for example [16, 10]) on the polygon defined by $\{A_0, A_1, A_2, A_3\}$.

We recall that a matrix is totally positive if all minors are nonnegative [1]. Then we obtain

THEOREM 4.1. *Suppose $-2 < \beta < 0$, $1 \le v := \frac{\lambda\beta}{\beta-1} \le 2$, and $\lambda \ge 1-\beta$. Then the matrix $\mathbf{S}$ given by (4.1) is totally positive. For each $v \notin [1, 2]$ there is a $\beta \in [-1, 0[$ such that $\mathbf{S}$ is not totally positive.*

Proof. The sequence of simple corner cuts corresponds to a factorization of $\mathbf{S}$ into a product of

7 matrices as follows:

$$\mathbf{S}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&\frac{1}{2}&\frac{1}{2}&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&\frac{v+\beta}{v-\beta}&\frac{-2\beta}{v-\beta}&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\frac{v-\beta}{2v}&\frac{v+\beta}{2v}&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$
$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&\frac{1}{v}&1-\frac{1}{v}\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\1-\frac{1}{v}&\frac{1}{v}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{v}{2}&1-\frac{v}{2}\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\1-\frac{v}{2}&\frac{v}{2}&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Since these matrices are bidiagonal and the entries are nonnegative for the indicated values of the parameters it is well known that each of the 7 matrices are totaly positive (see for example [10]). Since a product of totally positive matrices is totally positive we conclude that $\mathbf{S}$ is totally positive.

If $v\notin[1,2]$ then we can find $\beta\in[-1,0)$ such that $\mathbf{S}$ has at least one negative entry. Hence $\mathbf{S}$ is not totally positive for these v,β. ☐

4.2. The HC^1-Bernstein Basis. Let a,b be 2 real numbers with $a<b$ and let us define $h:=b-a$. Recall that the HC^1-Hermite basis $\{\phi_0,\psi_0,\phi_1,\psi_1\}$ on $I:=[a,b]$ forms a basis for the space $VC^1_{\alpha,\beta}(I)$ of all possible HC^1 interpolants on I. The *HC^1-Bernstein basis* $\{b_0,b_1,b_2,b_3\}$ on I are defined as in [17] from the Hermite basis on I by

$$b_0:=\phi_0-\frac{\lambda}{h}\psi_0,\quad b_1:=\frac{\lambda}{h}\psi_0,\quad b_2:=-\frac{\lambda}{h}\psi_1,\quad b_3:=\phi_1+\frac{\lambda}{h}\psi_1,\tag{4.3}$$

where $\lambda\geq 2$ is the parameter used to define the control points. These functions are clearly linearly independent and so, they form a basis for $VC^1_{\alpha,\beta}(I)$. The coefficients in terms of this basis are the control coefficients of f. This follows since

$$f:=f(a)\phi_0+p(a)\psi_0+f(b)\phi_1+p(b)\psi_1,\quad\Leftrightarrow\quad f=a_0b_0+a_1b_1+a_2b_2+a_3b_3,$$

where a_0,a_1,a_2,a_3 are the control coefficients of f on I given by (3.1).

We note that $b_j(0)=\delta_{j,0}$ and $b_j(1)=\delta_{j,3}$.

For certain values of the parameters the HC^1-Benstein basis is totally positive.

THEOREM 4.2. *Suppose $-2<\beta<0$, $1\leq v:=\frac{\lambda\beta}{\beta-1}\leq 2$, and $\lambda\geq 1-\beta$. Then the HC^1-Bernstein basis is totally positive.*

Proof. It is enough to prove the result for the interval $[0,1]$. Consider for some integers n,k with $n\geq 0$ and $0\leq k\leq 2^n-1$ the interval $I^n_k:=[k/2^n,(k+1)/2^n]$.

On I^n_k the HC^1-Hermite basis $\{\phi^n_{0,k},\psi^n_{0,k},\phi^n_{1,k},\psi^n_{1,k}\}$ can be expressed as

$$\begin{aligned}\phi^n_{0,k}(t)&=\phi_0(2^nt-k),\quad &\psi^n_{0,k}(t)&=2^{-n}\psi_0(2^nt-k),\\ \phi^n_{1,k}(t)&=\phi_1(2^nt-k),\quad &\psi^n_{1,k}(t)&=2^{-n}\psi_1(2^nt-k),\end{aligned}$$

where $\{\phi_0,\psi_0,\phi_1,\psi_1\}$ is the HC^1-Hermite basis on $[0,1]$. From (4.3) with $h:=2^{-n}$, it then follows that the HC^1-Bernstein basis $\{b^n_{4k},b^n_{4k+1},b^n_{4k+2},b^n_{4k+3}\}$ on I^n_k can be expressed in terms of the HC^1-Bernstein basis $\{b_0,b_1,b_2,b_3\}$ on $[0,1]$ as

$$b^n_{4k+j}(t)=\begin{cases}b_j(2^nt-k), & \text{if } t\in I^n_k \text{ and } j=0,1,2,3\\ 0 & \text{otherwise.}\end{cases}\tag{4.4}$$

We note that

$$b^n_{4k+j}(k/2^n) = \delta_{j,0}, \quad b^n_{4k+j}((k+1)/2^n) = \delta_{j,3} \quad \text{for} \quad j = 0,1,2,3. \tag{4.5}$$

Let $f \in C^1[0,1]$ be a HC^1-interpolant to some initial data. We can then write

$$f = \sum_{i=0}^{m} a_i^n b_i^n,$$

where $m := 4 \times 2^n - 1$ and where for $k = 0,\ldots,2^n - 1$ the numbers a^n_{4k}, a^n_{4k+1}, a^n_{4k+2}, a^n_{4k+3} are the control points of f on I^n_k. In vector form, we have $f = b^n a^n$ where $b^n = (b^n_0,\ldots,b^n_m)$ is a row vector and $a^n = (a^n_0,\ldots,a^n_m)^T$ a column vector. Note that b^n is a vector of linearly independent functions on $[0,1]$. They span a space containing $VC^1_{\alpha,\beta}[0,1]$ as a 4-dimensional subspace. On level $n+1$ we have $f = b^{n+1}a^{n+1}$, where from Proposition 3.1 it follows that $a^{n+1} = \mathbf{A}_n a^n$ for some matrix $\mathbf{A}_n$. The matrix $\mathbf{A}_n$ is a block diagonal with 2^n diagonal blocks $\hat{\mathbf{S}}$ of order 8×4. Indeed, $\hat{\mathbf{S}}$ is obtained from the matrix $\mathbf{S}$ in (3.4) by adding a copy of row 4 as a new row 5. But then $f = b^{n+1}a^{n+1} = b^{n+1}\mathbf{A}_n a^n = b^n a^n$ and by linear independence, it follows that $b^n = b^{n+1}\mathbf{A}_n$. Thus we obtain

$$b^0 = b^n \mathbf{A}_{n-1}\cdots\mathbf{A}_0, \quad n \geq 1. \tag{4.6}$$

For distinct points $y_0,\ldots,y_p$ and functions $f_0,\ldots,f_q$ defined on the y's, we use the standard notation

$$M\begin{bmatrix} y_0,\ldots,y_p \\ f_0,\ldots,f_q \end{bmatrix} := \begin{bmatrix} f_0(y_0) & \cdots & f_q(y_0) \\ \vdots & & \vdots \\ f_0(y_p) & \cdots & f_q(y_p) \end{bmatrix}$$

for a collocation matrix of order $(p+1)\times(q+1)$. In order to show total positivity of $b = b^0$ we choose $0 \leq x_0 < x_1 < x_2 < x_3 \leq 1$ and consider the collocation matrix $M\begin{bmatrix} x_0,x_1,x_2,x_3 \\ b_0,b_1,b_2,b_3 \end{bmatrix}$. From (4.6) we immediatly obtain

$$M\begin{bmatrix} x_0,\ldots,x_3 \\ b_0,\ldots,b_3 \end{bmatrix} = M\begin{bmatrix} x_0,\ldots,x_3 \\ b^n_0,\ldots,b^n_m \end{bmatrix}\mathbf{A}_{n-1}\cdots\mathbf{A}_0, \quad n \geq 1. \tag{4.7}$$

Since the matrix $\mathbf{S}$ is totally positive, it follows that $\hat{\mathbf{S}}$ and hence each $\mathbf{A}_k$ is totally positive. We now show that the first matrix on the right of (4.7) is totally positive provided $x_j \in \mathcal{P}_n$ for $j = 0,1,2,3$. For this, with $m = 2^{n-1} - 1$, we consider the bigger matrix

$$\mathbf{B} = M\begin{bmatrix} y_0,\ldots,y_{m+1} \\ b^n_0,\ldots,b^n_m \end{bmatrix}$$

using all points $y_i = i/2^n$, $i = 0,1,\ldots,2^n$ in $\mathcal{P}_n$. From (4.5) it follows that $b_{4k-1}(y_k) = 1$ for $k = 1,\ldots,2^n$, $b_{4k}(y_k) = 1$ for $k = 0,\ldots,2^n - 1$ and $b^n_i(y_j) = 0$ otherwise. Thus the columns of $\mathbf{B}$ have the following form

$$\mathbf{B} = [e_1, 0, 0, e_2, e_2, 0, 0, e_3, e_3, 0, 0, e_4 \ldots, e_m, 0, 0, e_{m+1}],$$

where $e_j = (\delta_{i,j})^m_{i=0}$ is the jth unit vector in $\mathbb{R}^{m+1}$. From this explicit form we see that $\mathbf{B}$ is totally positive since each nonzero minor must be the determinant of the identity matrix. But then all matrices on the right in (4.7) are totally positive and we conclude that $M\begin{bmatrix} x_0,\ldots,x_3 \\ b_0,\ldots,b_3 \end{bmatrix}$ is totally positive provided $x_j \in \mathcal{P}_n$ for $j = 0,1,2,3$. Since $\cup\mathcal{P}_n$ is dense in $[0,1]$ we conclude that the HC^1-Bernstein basis is totally positive. ∎

Corollary 4.3. *For $p \geq 0$ and $m = 4 \cdot 2^p - 1$, the basis $b^p = (b_0^p, \ldots, b_m^p)$ for the space span(b^p) is totally positive on* $[0, 1]$.

Proof. Instead of (4.6) we use for $n > p$ the equation

$$b^p = b^n \mathbf{A}_{n-1} \cdots \mathbf{A}_p.$$

The argument now proceeds as in the proof of Theorem 4.2 replacing $x_0, \ldots, x_3$ by suitable $x_0, \ldots, x_m$. $\square$

It is well known that total positivity of the HC^1 Bernstein basis on $[0, 1]$ implies that the HC^1-interpolant f inherits properties of the control polygon P^0 defined by $\{a_0, a_1, a_2, a_3\}$, see for example ([10]). In particular if P_0 is positive (monotone, convex) then f is positive (monotone, convex). We can use this to generalize Theorem 4 in ([17]).

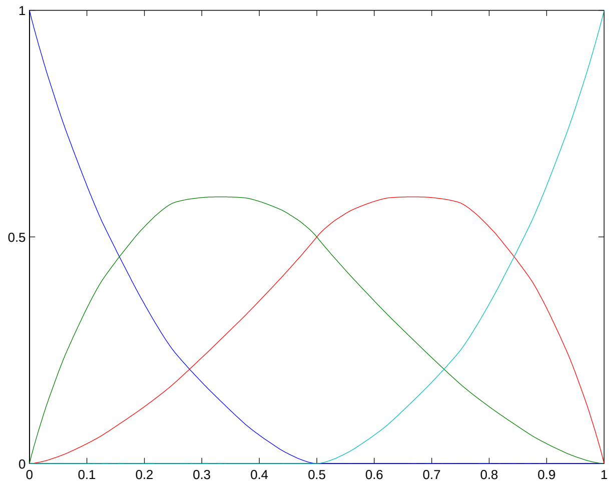

Fig. 4.2. *Bernstein basis,* $\beta = -3/5$, $\alpha = -3/32, \lambda = 16/3$.

Corollary 4.4. *Let b_0, b_1, b_2, b_3 be the HC^1 Bernstein basis on $[0, 1]$ given by (4.3) with $\lambda = v(\beta - 1)/\beta > 2$. Suppose also $\alpha = \frac{\beta}{4(1-\beta)}$, $-1 \leq \beta < 0$ and $1 \leq v \leq 2$. Then*

1. *b_0 is nonnegative, decreasing, and convex on $[0, 1]$. If $v = 2$ then $b_0(t) = 0$ for $t \in [1/2, 1]$.*
2. *b_1 is nonnegative and concave on $[0, 1/2]$ and nonnegative, decreasing and convex on $[1/2, 1]$*
3. *b_2 is nonnegative, increasing and convex on $[0, 1/2]$ and nonnegative and concave on $[1/2, 1]$.*
4. *b_3 is nonnegative, increasing, and convex on $[0, 1]$. If $v = 2$ then $b_3(t) = 0$ for $t \in [0, 1/2]$.*
5. *$\sum_{j=0}^{3} b_j(t) = 1$ for $t \in [0, 1]$*

Proof. From (4.3) it follows that the control points of the function b_j is the jth unit vector e_{j+1} for $j = 0, 1, 2, 3$. Thus nonnegativity of b_j follows from the nonnegativity of e_{j+1} for $j = 0, 1, 2, 3$. Moreover the monotonicity and convexity properties of b_0 and b_3 follow. For the remaining properties of b_1 and b_2, we carry out one subdivision, then the proof is similar.

The refined points are given as the columns of the matrix $\mathbf{S}$ given by (4.1). When $v = 2$ the first column is given by $[1, 1/2, 0, 0, 0, 0, 0, 0]$. Since the last four entries are zero it follows that $b_0(t) = 0$ for $t \in [1/2, 1]$. Similarly $b_3(t) = 0$ for $t \in [0, 1/2]$.

The interpolation of the constant function $f = 1$ with $p = f' = 0$ gives $a_0 = a_1 = a_2 = a_3 = 1$ in (3.1) so that 5. holds. $\square$

5. Algorithms for local shape constraints. We base shape preserving algorithms on the extended quadratic spline case given by $\alpha = \frac{\beta}{4(1-\beta)}$. The control point subdivision matrix for this case is given by (4.1), where we have both β and λ as free parameters. The matrix simplifies when

$v = \frac{\beta\lambda}{\beta-1} = 2$ and we will use this one parameter family of schemes in our algorithms. Using the parameter λ to control the shape we thus have

$$\alpha = \frac{\beta}{4(1-\beta)} = -\frac{1}{2\lambda}, \quad \beta = \frac{2}{2-\lambda}. \tag{5.1}$$

We restrict our attention to $\lambda \geq 4$. We then have $\beta \in [-1, 0)$ and both algorithms HC^1 and SC^1 are convergent. In the limit when $n \to \infty$ we obtain a function $f \in C^1(I)$. This function is the quadratic spline interpolant with a knot at the midpoint of I when $\lambda = 4$, while $p = f'$ is Hölder continuous on I with exponent

$$\log_2\left(1 + \frac{1}{\lambda - 3}\right) \approx \frac{1.44}{\lambda - 3}, \quad \lambda \to \infty.$$

Thus the derivative becomes less regular when λ increases, but it is always C^1.

Given $s \geq 1$, points $a_j^0 = a_j \in \mathbb{R}^s$ for $j = 0, 1, 2, 3$, and $\lambda \geq 4$, the following algorithm computes sequences $\{a^n\}$ of control coefficients $a^n = (a_0^n, a_1^n, \ldots, a_{3\times 2^n}^n)$ in $\mathbb{R}^s$.

Algorithm 5.1 (CC^1).

1. $\beta = 2/(2 - \lambda)$;
2. *For* $n = 0, 1, 2, 3, \ldots$
   For $i = 0, 1, \ldots 2^n - 1$
   (a) $a_{6i}^{n+1} = a_{3i}^n$;
   (b) $a_{6i+1}^{n+1} = \frac{1}{2}(a_{3i}^n + a_{3i+1}^n)$;
   (c) $a_{6i+2}^{n+1} = (\frac{1}{2} - \frac{\beta}{4})a_{3i+1}^n + (\frac{1}{2} + \frac{\beta}{4})a_{3i+2}^n$;
   (d) $a_{6i+3}^{n+1} = \frac{1}{2}(a_{3i+1}^n + a_{3i+2}^n)$;
   (e) $a_{6i+4}^{n+1} = (\frac{1}{2} + \frac{\beta}{4})a_{3i+1}^n + (\frac{1}{2} - \frac{\beta}{4})a_{3i+2}^n$;
   (f) $a_{6i+5}^{n+1} = \frac{1}{2}(a_{3i+2}^n + a_{3i+3}^n)$;
   $a_{3\cdot 2^{n+1}} = a_{3\cdot 2^n}$;

The control points corresponding to the computed control coefficients converges to a C^1-curve. More specifically, pick any finite closed interval $[a, b]$ and define $h_n := (b - a)/2^n$ and $t_k^n := a + kh_n$ for $k = 0, \ldots, 2^n$, $n \geq 0$. By Theorem 3.4 the computed control points converge uniformly to a C^1-curve $f : [a, b] \to \mathbb{R}^s$. Moreover,

$$\begin{aligned} a_{3i}^n &= f(t_i^n), \quad i = 0, \ldots, 2^n, \\ a_{3i+1}^n - a_{3i}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 0, \ldots, 2^n - 1, \\ a_{3i}^n - a_{3i-1}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 1, \ldots, 2^n. \end{aligned}$$

We now discuss shape preservation in the scalar case $s = 1$ in more detail. We start by noting that if the initial control polygon is nonnegative (respectively increasing, convex) on an interval $I = [a, b]$, then the HC^1-interpolant computed in Algorithm 5.1 will be nonnegative (respectively increasing, convex) on the same interval I. This follows from the total positivity of the Bernstein basis. In addition to total positivity the main tool will be Corollary 2.2 which says that the p-values of the interpolant are located on the piecewise linear curve connecting the three points $(a, p(a))$, $(\frac{a+b}{2}, p(\frac{a+b}{2}))$, $(b, p(b))$.

5.1. Nonnegative Interpolants. We already remarked that if the initial control coefficients are nonnegative then the HC^1-interpolant will be nonnegative. Notice that the converse is false. For example, the HC^1-interpolant to the function f given on $[0, 1]$ by $f(x) := 16(x - 1/4)^2$ and using $\lambda = 4$ is f itself. Note that f is nonnegative, but the initial control coefficient $a_1 = -1$ is negative.

To give an algorithm for constructing a nonnegative interpolant we assume that

$$f(a) \geq 0, \; f(b) \geq 0, \; p(a) \geq 0 \text{ if } f(a) = 0, \text{ and } p(b) \leq 0 \text{ if } f(b) = 0. \tag{5.2}$$

Under these weak assumptions nonnegative initial control coefficients $a_0, \dots, a_3$ can always be obtained by choosing λ sufficiently large. Indeed, since $a_0 = f(a) \ge 0$ and $a_3 = f(b) \ge 0$ we only need to make sure that $a_1 = f(a) + hp(a)/\lambda \ge 0$ and $a_2 = f(b) - hp(b)/\lambda \ge 0$. If $f(a) = 0$ then $p(a) \ge 0$ and $a_1 \ge 0$ whenever $\lambda > 0$. Similarly $a_2 \ge 0$ if $f(b) = 0$. But then we can choose $\lambda = 4$ except possibly in the two cases $f(a) > 0$, $p(a) < 0$ and $f(b) > 0$, $p(b) > 0$. If (5.2) holds then the following algorithm will compute a nonnegative HC^1-interpolant on $[a, b]$.

Algorithm 5.2 (Nonnegative Interpolant).

1. *Compute* λ
   (a) $\lambda = 4$;
   (b) *if* $(f(a) > 0)$ & $(p(a) < 0)$ *then* $\lambda = \max(\lambda, -hp(a)/f(a))$;
   (c) *if* $(f(b) > 0)$ & $(p(b) > 0)$ *then* $\lambda = \max(\lambda, hp(b)/f(b))$;
2. *Compute initial control coefficients using* (3.1).
3. *Apply Algorithm 5.1 or Algorithm* HC^1 *with* $\alpha = -\frac{1}{2\lambda}$, $\quad \beta = \frac{2}{2-\lambda}$.

5.2. Monotone interpolants. The monotonicity of the HC^1-interpolant is completely determined by the monotonicity of the initial control polygon. If f is decreasing then $-f$ is increasing and we restrict our discussion to increasing interpolants.

Proposition 5.3. *Suppose that the parameters are chosen according to* (5.1). *Then the* HC^1*-interpolant* f *is nondecreasing on an interval* $I = [a, b]$ *if and only if the control polygon on* I *is nondecreasing.*

Proof. By Theorem 4.2 the Bernstein basis is totally positive and it follows that the HC^1-interpolant is nondecreasing if the control polygon is nondecreasing , see [10]. Conversely, suppose the HC^1-interpolant f is nondecreasing. Since $\beta = 2/(2 - \lambda)$, we obtain from (2.1)

$$p(\frac{a+b}{2}) = \frac{1}{\lambda - 2}\left(\lambda\frac{f(b) - f(a)}{h} - (p(a) + p(b))\right). \tag{5.3}$$

From (3.1), we then find

$$a_1 - a_0 = \frac{h}{\lambda}p(a), \quad a_2 - a_1 = \frac{\lambda - 2}{\lambda}hp(\frac{a+b}{2}), \quad a_3 - a_2 = \frac{h}{\lambda}p(b). \tag{5.4}$$

Now $p \ge 0$ at all points if f is nondecreasing. It follows that the control coefficients, and hence the control polygon is nondecreasing. □

Consider next the case of a strictly increasing interpolant.

Proposition 5.4. *Suppose that the parameters are chosen according to* (5.1) *and that the* HC^1*-interpolant* f *is nondecreasing on an interval* $I = [a, b]$. *Then* f *is strictly increasing on* $[a, b]$ *if and only if the two middle control coefficients on* I *satisfy* $a_2 > a_1$.

Proof. Since f is nondecreasing, we have $p(a) \ge 0$, $p(\frac{a+b}{2}) \ge 0$ and $p(b) \ge 0$. By Corollary 2.2, it follows that f is strictly increasing on $[a, b]$ if and only if $p(\frac{a+b}{2}) > 0$. By (5.4), this happens if and only if $a_2 > a_1$. □

To give an algorithm to construct a nondecreasing interpolant we assume that

$$f(a) \le f(b),\ p(a) \ge 0,\ p(b) \ge 0 \text{ and } p(a) = p(b) = 0 \text{ if } f(a) = f(b). \tag{5.5}$$

In the latter case the HC^1-interpolant is constant and we can set $\lambda = 4$.

Suppose $f(b) > f(a)$. With $h := b - a$ we then have

$$a_0 = f(a) \le a_1 = f(a) + \frac{h}{\lambda}p(a) \le a_2 = f(b) - \frac{h}{\lambda}p(b) \le a_3 = f(b)$$

provided

$$a_2 - a_1 = f(b) - f(a) - \frac{h}{\lambda}\big(p(b) + p(a)\big) \ge 0$$

or

$$\lambda \geq \frac{(p(a)+p(b))h}{f(b)-f(a)}. \tag{5.6}$$

If (5.5) holds then the following algorithm will compute a nondecreasing HC^1-interpolant on $[a,b]$. It will be strictly increasing if $f(b) > f(a)$ and (5.6) holds with strict inequality.

ALGORITHM 5.5 (Nondecreasing- or Strictly Increasing Interpolant).

1. *Compute* λ
   (a) $\lambda = 4$;
   (b) *If* $f(a) < f(b)$ *then*
      i. $\lambda_1 \geq \frac{(p(a)+p(b))h}{f(b)-f(a)}$
      ii. $\lambda = \max(4, \lambda_1)$
2. *Compute initial control coefficients using* (3.1).
3. *Apply Algorithm 5.1 or Algorithm* HC^1 *with* $\alpha = -\frac{1}{2\lambda}$, $\quad \beta = \frac{2}{2-\lambda}$.

Note that if the initial control points are located on a straight line then the HC^1-interpolant is the line segment connecting the first and last control point. For if the initial control points are located on a straight line then

$$\frac{\lambda}{h}(a_1 - a_0) = \frac{\lambda}{(\lambda-2)h}(a_2 - a_1) = \frac{\lambda}{h}(a_3 - a_2)$$

and by (5.4) the three slopes $p(a), p(\frac{a+b}{2}), p(b)$ are all equal. By Corollary 2.2, all slopes are equal and the function f is a straight line.

In Figure 5.1 we interpolate three sets of data on $[0,1]$. In all cases $f(0) = -1$ and $f(1) = 1$. In the first case, with $p(0) = 3$ and $p(1) = 4$ we find $\frac{p(0)+p(1)}{f(1)-f(0)} = 7/2 < 4$. Suppose in Algorithm 5.5 we choose $7/2 \leq \lambda_1 \leq 4$ in Statement $(b)i.$ and apply Algorithm 5.1 with $\lambda = 4$. Then the HC^1-interpolant is the quadratic spline and it is strictly increasing since $\lambda > 7/2$. In the two other cases we use $p(0) = 8$ and $p(1) = 4$ giving $\frac{p(0)+p(1)}{f(1)-f(0)} = 6$. With $\lambda = 6$ we have $p(1/2) = 0$ and the interpolant is increasing, but not strictly increasing. We obtain a strictly increasing interpolant by using $\lambda = 10$. Note that choosing a bigger λ decreases the regularity of the interpolant. In both cases the first derivative is Hölder continuous, but the exponent is $\log_2(4/3) \approx 0.415$ when $\lambda = 6$ and $\log_2(4/3) \approx 0.193$ when $\lambda = 10$.

5.3. Convex interpolants. The convexity of the HC^1-interpolant is also completely determined by the convexity of the initial control polygon.

PROPOSITION 5.6. *Suppose that the parameters are chosen according to* (5.1). *Then f is convex (concave) on an interval $I = [a,b]$ if and only if the control polygon on I is convex (concave).*

Proof. Again by total positivity of the Bernstein basis the HC^1-interpolant is convex (concave) if the control polygon is convex (concave), see [10]. Conversely, suppose the HC^1-interpolant f is convex (concave). Now the control polygon is convex if and only if the conditions

$$\frac{a_1 - a_0}{h/\lambda} \leq \frac{a_2 - a_1}{h - 2h/\lambda} \leq \frac{a_3 - a_2}{h/\lambda}$$

hold. But from (5.4) we find

$$\frac{a_1 - a_0}{h/\lambda} = p(a), \quad \frac{a_2 - a_1}{h - 2h/\lambda} = p(\frac{a+b}{2}), \quad \frac{a_3 - a_2}{h/\lambda} = p(b).$$

Since f is convex (concave) the function p is nondecreasing (nonincreasing) and hence the control polygon is convex(concave). □

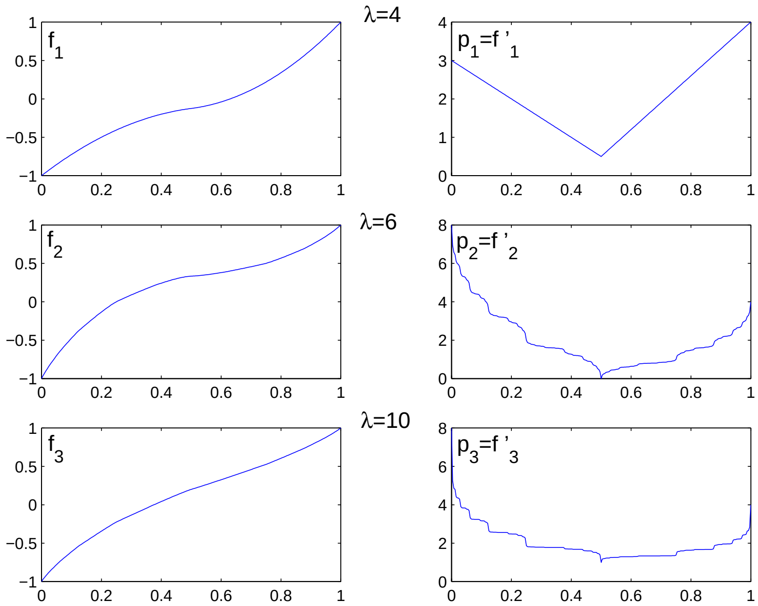

FIG. 5.1. *Monotone interpolants*

To give an algorithm for constructing a convex (concave) HC^1-interpolant on an interval $I = [a, b]$ we first assume that

$$p(a) < \frac{f(b) - f(a)}{h} < p(b) \quad \left(p(a) > \frac{f(b) - f(a)}{h} > p(b)\right), \tag{5.7}$$

where $h := b - a$. We define

$$\lambda_1 := \frac{p(b) - p(a)}{p(b) - \frac{f(b)-f(a)}{h}}, \quad \lambda_2 := \frac{p(b) - p(a)}{\frac{f(b)-f(a)}{h} - p(a)} \tag{5.8}$$

and note that the tangents

$$t_c(x) := f(a) + (x - a)p(a), \quad t_d(x) := f(b) + (x - b)p(b)$$

of f at a and b intersect at the point$(\bar{x}, \bar{y})$ given by

$$\frac{\bar{x} - a}{h} = \frac{1}{\lambda_1}, \quad \text{and} \quad \frac{b - \bar{x}}{h} = \frac{1}{\lambda_2}.$$

Moreover, the hypothesis (5.7) is equivalent to $a < \bar{x} < b$.

Under the assumption

$$p(a) \le \frac{f(b) - f(a)}{h} \le p(b) \quad \left(p(a) \ge \frac{f(b) - f(a)}{h} \ge p(b)\right) \tag{5.9}$$

the following algorithm will compute a convex (concave) interpolant.

ALGORITHM 5.7 (Convex or Concave interpolant).

1. *(a)* *If* $p(a) = \frac{f(b)-f(a)}{h} \neq p(b)$*, choose* $\lambda \ge \max(4, \lambda_1)$
   (b) *If* $p(a) \neq \frac{f(b)-f(a)}{h} = p(b)$*, choose* $\lambda \ge \max(4, \lambda_2)$

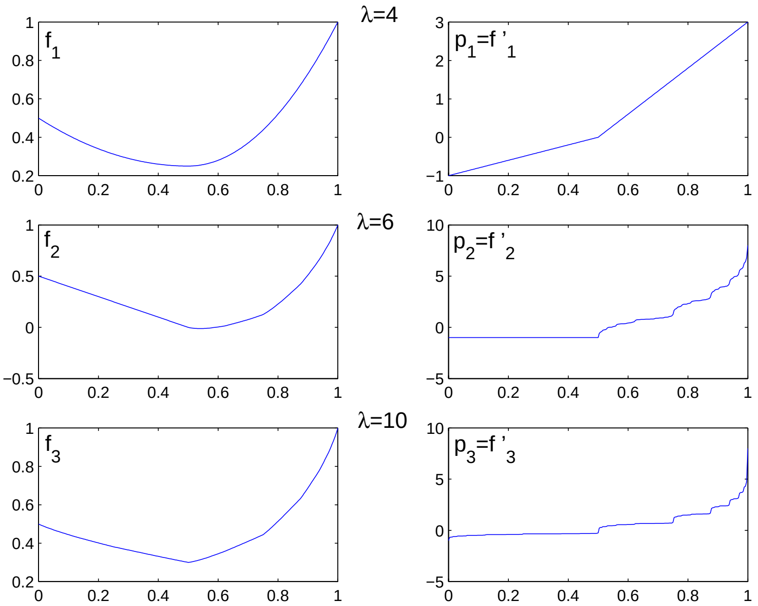

FIG. 5.2. *Convex interpolants*

(c) *If* $p(a) \neq \frac{f(b)-f(a)}{h} \neq p(b)$, *choose* $\lambda \geq \max(4, \lambda_1, \lambda_2)$

2. *Compute initial control points using* (3.1)
3. *Apply Algorithm 5.1 or Algorithm* HC^1 *with* $\alpha = -\frac{1}{2\lambda}$, $\quad \beta = \frac{2}{2-\lambda}$.

In Figure 5.2, we have interpolated three sets of data on $[0, 1]$. In all cases $f(0) = 0.5$ and $f(1) = 1$.

In the first case, $p(0) = -1$ and $p(1) = 3$ so that $\lambda_1 = 8/5$ and $\lambda_2 = 8/3$. Then $\max(4, \lambda_1, \lambda_2) = 4$ and we have chosen $\lambda = 4$. In this case, the interpolant is the quadratic spline.

In the two other cases $p(0) = -1$ and $p(1) = 8$ so that $\lambda_1 = 18/5$ and $\lambda_2 = 6$. Then $\max(4, \lambda_1, \lambda_2) = 6$. With $\lambda = 6$ we have $p = -1$ on $[0, 1/2]$, while we obtain a strictly convex interpolant by using $\lambda = 10$. Recall that choosing a bigger λ decreases the regularity of the interpolant.

6. Example. Given data (t_i, y_i, y_i') for $i = 1, \ldots, n$, where $t_1 < \cdots < t_n$ and the y's are real numbers. We look for a function $f \in C^1([t_1, t_n])$ that satisfies

$$f(t_i) = y_i, \;\; f'(t_i) = y_i' \;\; \text{for } i = 1, \ldots, n. \tag{6.1}$$

In addition we would like f to be positive, monotone, linear, or convex on some or all of the subintervals $I_i = [t_i, t_{i+1}], i = 1, \ldots n-1$. We assume that

(P) (5.2) holds for the subintervals where we want nonnegativity or positivity.

(M) (5.5) holds for the subintervals where we want a nondecreasing or a strictly increasing interpolant.

(L) $y_i' = y_{i+1}' = \frac{y_{i+1}-y_i}{t_{i+1}-t_i}$ for the subintervals where the interpolant should be linear.

(C) (5.9) holds for the subintervals where the interpolant should be convex or concave.

We also require that the given data is consistent with these shape requirements. We can compute f locally by applying the HC^1-algorithm with parameters given by (5.1) on each subinterval $I_i = [t_i, t_{i+1}], i = 1, \ldots n-1$ using initial data $f(t_i) = y_i, f(t_{i+1}) = y_{i+1}$, $p(t_i) = y_i'$ and $p(t_{i+1}) = y_{i+1}'$. We obtain C^1-convergence and the desired shape locally by choosing the parameter λ_i for the interval I_i sufficiently large.

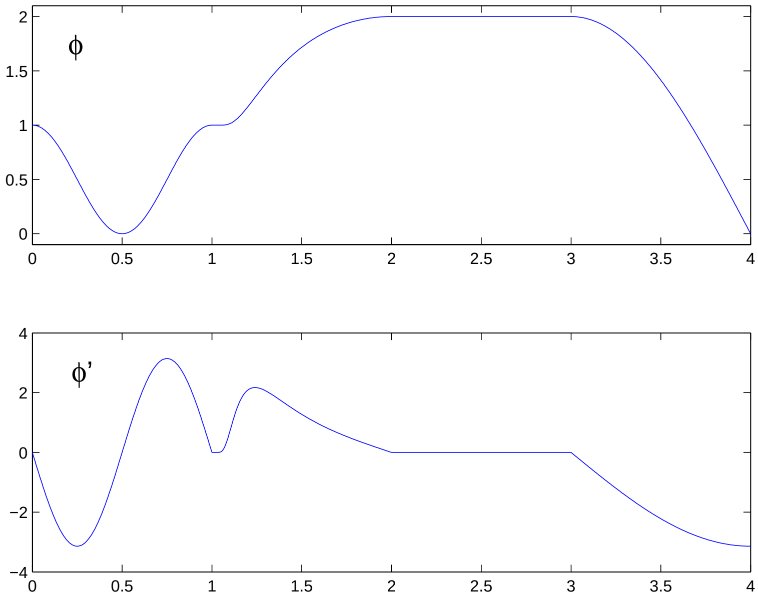

FIG. 5.3. *The function ϕ and its derivative*

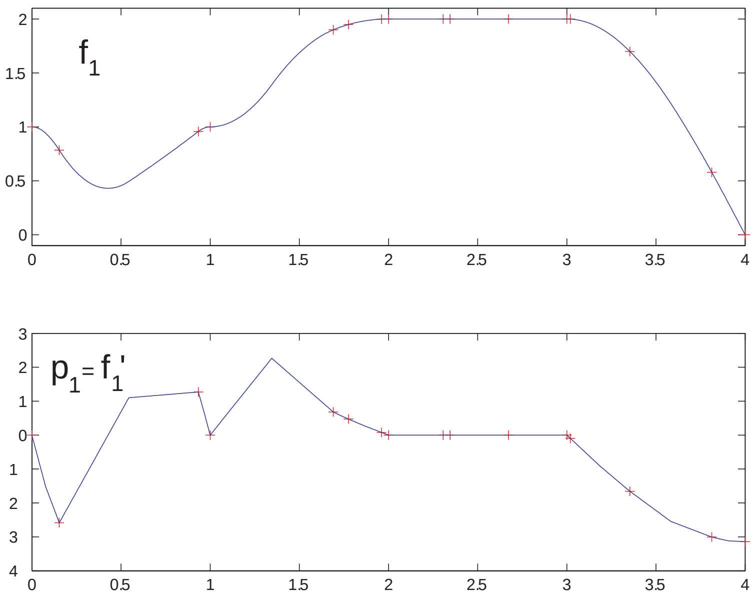

FIG. 6.1. *Interpolation with exact derivatives*

Consider now (6.1) for the example illustrated in Figure 1.1. The data are sampled from the function $\phi \in C^1([0,4])$ given by

$$\phi(t) = \begin{cases} \frac{1}{2}\sin(2\pi t + \pi/2) + \frac{1}{2} & , \quad 0 \le t \le 1, \\ 1 + \exp(-\frac{1}{1-(t-2)^2} + 1) & , \quad 1 < t \le 2, \\ 2 & , \quad 2 < t \le 3, \\ 2\cos(\pi \frac{t-3}{2}) & , \quad 3 \le t \le 4. \end{cases} \tag{6.2}$$

The function and its first derivative are displayed in Figure 5.3 and it can be shown that ϕ is positive on $[0,1]$, strictly increasing on $[1,2]$, constant on $[2,3]$ and concave on $[3,4]$. Given n and let $(t_1, \ldots, t_n)$ be a partition of $[0, 4]$. The points $(t_2, \ldots, t_{n-1})$ are chosen randomly except that $1, 2, 3$ are among them. In the example, we used $t_1 = 0$, $t_{n_1} = t_5 = 1$, $t_{n_2} = t_9 = 2$, $t_{n_3} = t_{13} = 3$ and $t_n = t_{17} = 4$. We want an interpolant f which is positive on $[t_1, t_{n_1}] = [0, 1]$, strictly increasing on $[t_{n_1}, t_{n_2}] = [1, 2]$, constant on $[t_{n_2}, t_{n_3} = [2, 3]$ and concave on $[t_{n_3}, t_n] = [3, 4]$.

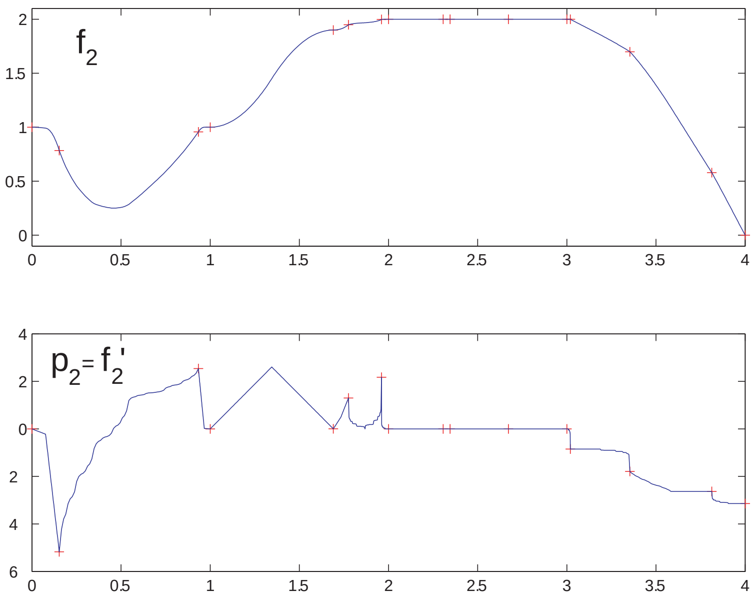

FIG. 6.2. *Interpolation with modified derivatives*

In the first test we use $y_i = \phi(t_i)$ and exact derivatives $y'_i = \phi'(t_i), i = 1, \ldots, n$. In this case all λ's become equal to 4 and the quadratic spline interpolant f_1 does the job. Plots of this function and its first derivative are shown in Figure 6. The first derivative appears continuous and piecewise linear.

For the second test shown in Figure 6, we kept the previous data t_i and $y_i = \phi(t_i)$ for $i = 1, \ldots, n = 17$, but we used inexact derivatives given by crosses in the lower part of the figure. However the derivatives were chosen so that the relevant requirement (P),(M), (L), and (C) above are satisfied on each subinterval $[t_i, t_{i+1}]$. We obtain a C^1-interpolant f_2 satisfying the required shape constraints. The computed values of λ_i are successively
$(4, 5.1425, 4, 4, 4, 12.8631, 55.8239, 4, 4, 4, 4, 17.6767, 20.0216, 4.4087, 11.3544)$. These are the smallest values on each interval. Any larger value of λ_i is possible without loosing the shape, but then the curve is less regular (smaller Hölder exponent) on the corresponding interval. This example shows that we can obtain a desired shape even with more or less random derivative values.

7. Conclusion. We have shown that the Hermite subdivision scheme introduced by Merrien in 1992 has many desirable properties. It gives a C^1 limit curve for a wide range of parameters. A one parameter subfamily called the Extended Quadratic Spline-scheme is particularly interesting. This family can be formulated as a scheme, the SC^1 algorithm, with a totally positive subdivision matrix. When applied in a piecewise fashion its local nature makes it easy to control the final shape of the subdivision curve. In many cases a desired shape can be obtained even without accurate derivative estimates.

The SC^1 algorithm can also be used in the parametric case, but a discussion of this will be deferred to a future paper. We also defer the construction of interpolating C^1 surfaces with shape preserving properties.

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