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HERMITE SUBDIVISION WITH SHAPE CONSTRAINTS ON A RECTANGULAR MESH*

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Abstract.

In 1999, Dubuc and Merrien introduced a Hermite subdivision scheme which gives C^1-interpolants on a rectangular mesh. In this paper a two parameter version of this scheme is analyzed, and C^1-convergence is proved for a range of the two parameters. By introducing a control grid the parameters in the scheme can be chosen so that the interpolant inherits positivity and/or directional monotonicity from the initial data. Several examples are given showing that a desired shape can be achieved even if only very crude estimates for the initial slopes are used.

AMS subject classification (2000): 65D05, 65D17.

Key words: interpolation, subdivision, Hermite data, rectangular mesh, positivity, monotonicity.

1 Introduction.

Subdivision is a technique for constructing smooth curves or surfaces out of a sequence of successive refinements of polygons, or grids see [3]. Subdivision has found applications in areas such as geometric design [10, 21], and in computer games and animation [6]. Subdivision schemes can be of Lagrange type or Hermite type. In this last case derivatives are also used. This can be desirable since a Hermite scheme can be made more local, making it easier to obtain a desired shape. Moreover, as our examples show, we can achieve a required shape using only very crude estimates for the derivatives. For some classical methods for bimonotone interpolation on a rectangular grid see [1, 2, 4, 5].

The first Hermite scheme for univariate functions was introduced in [15]. This method has smoothness C^1 and we refer to it as the HC^1-scheme. A notion of control points for two subfamilies of the HC^1-scheme were introduced in [18]. In [14] some further studies of the HC^1-scheme where carried out. The calculation of values and derivatives was separated and this made it possible to

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simplify some of the proofs in [17]. It was also shown that a geometric formulation of the scheme has a totally positive transformation matrix, and algorithms for constructing curves satisfying local positivity, monotonicity, and convexity constraints were given and tested. For more references to Hermite subdivision see [8, 9, 12, 13, 16, 22].

In [11, 18] Hermite subdivision was studied on a rectangular mesh using tensor products of the HC^1-scheme and its control points. An algorithm for achieving a bimonotone interpolant was given.

A disadvantage using the tensor product construction is that mixed partial derivatives $\partial^2 f/\partial x \partial y$ is required as input data. In this paper we consider an alternative method the HRC^1-algorithm, where these mixed partial derivatives are not required. This scheme was introduced in [7]. It is a generalization of a C^1-quadratic finite element on a quadrilateral mesh ([20]). To describe the HRC^1-algorithm we start with values and gradients at the vertices of a rectangular grid G in the plane. The algorithm is applied to each rectangle R in turn by a local process. We divide R into 4 rectangles by connecting midpoints of opposite edges and then compute values and derivatives at the vertices of the 4 sub-rectangles. Repeating this on each sub-rectangle we obtain in the limit a function defined on a dense subset of R. The scheme is interpolatory, i.e. it retains the values at the vertices of the current rectangular grid. Moreover the value on an edge E of R only depends on the length of E and on the values of f and its derivatives at the endpoints of E. This makes it possible to obtain a global smooth surface by gluing together HRC^1-interpolants on neighboring sub-rectangles.

Our paper can be detailed as follows. In Section 2, we first recall the HRC^1-algorithm and some of its properties which were proved in [7]. We consider a simplified version of the scheme using only two parameters α and β. We show that this version simplifies further if we choose $\alpha = \beta/(4(1-\beta))$.

In Section 3 we show C^1-convergence of the HRC^1-algorithm for a range of the parameters α and β. This extends results in [7] where C^1-convergence was only shown for $\alpha = -1/8$. We also show Hölder continuity of the first order partial derivatives.

In Section 4 we define a control grid thereby giving a geometric formulation of the HRC^1-algorithm. This formulation is used in Section 5 to show how local shape constraints can be achieved in the limit function. We give several examples involving positivity and directional monotonicity constraints. We also show that a convexity preserving HRC^1-interpolant cannot be obtained in general.

2 Description of the algorithm HRC^1.

We let $R := [a, b] \times [c, d]$ be a given rectangle. The algorithm HRC^1 which gives a C^1 Hermite interpolant on R was proposed by Dubuc and Merrien [7]. The goal is to construct a bivariate function f and its first partial derivatives $p := f_x$, $q := f_y$ on R in such a way that f, p, q are continuous.

The HRC^1-algorithm can be formulated as follows. We start with Hermite data $f^0_{i,j}$, $p^0_{i,j}$, $q^0_{i,j}$ for $i,j = 0,1$ at the corners of the rectangle $[s^0_0, s^0_1] \times [t^0_0, t^0_1] := [a,b] \times [c,d]$. For $n = 0,1,2,\dots$, let us denote by $\mathcal{P}_n$ the regular partition of $[a,b]$ into 2^n subintervals and by $\mathcal{Q}_n$ the similar regular partition of $[c,d]$. Also let $\mathcal{P} := \cup_{n\in\mathbb{N}} \mathcal{P}_n$ and $\mathcal{Q} := \cup_{n\in\mathbb{N}} \mathcal{Q}_n$. For $n = 0,1,2,\dots$ the points in the partitions are denoted by $s^n_i := a + ih_n$ and $t^n_j := c + jk_n$, for $i,j = 0,\dots,2^n$, where $h_n := 2^{-n}(b-a)$ and $k_n := 2^{-n}(d-c)$. What we compute at the grid points (s^n_i, t^n_j) can be viewed either as point sequences $\{f^n_{ij}\}$, $\{p^n_{ij}\}$, $\{q^n_{ij}\}$ or as functions $f,p,q : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}$ defined by $f(s^n_i, t^n_j) := f^n_{ij}$, $p(s^n_i, t^n_j) := p^n_{ij}$, and $q(s^n_i, t^n_j) := q^n_{ij}$. We will use both the sequence point of view and the function point of view.

To define f, p and q on $\mathcal{P}\times\mathcal{Q}$, we proceed by induction on n. For $n = 0,1,2,\dots$ suppose we have computed $\{f^n_{i,j}\}$, $\{p^n_{i,j}\}$, and $\{q^n_{i,j}\}$ on the grid $\mathcal{P}_n \times \mathcal{Q}_n$. We set $h_n := 2^{-n}(b-a)$, $k_n := 2^{-n}(d-c)$ and compute $f^{n+1}_{i,j}$, $p^{n+1}_{i,j}$, and $q^{n+1}_{i,j}$ on the grid $\mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}$ as follows:

$$\boxed{\begin{array}{l} \text{for } i = 2^n : -1 : 0, \ \text{for } j = 2^n : -1 : 0 \\ \quad f^{n+1}_{2i,2j} := f^n_{i,j}, \quad p^{n+1}_{2i,2j} := p^n_{i,j}, \quad q^{n+1}_{2i,2j} := q^n_{i,j}, \end{array}} \tag{2.1}$$

$$\boxed{\begin{array}{l} \text{for } i = 0 : 2^n - 1, \quad \text{for } j = 0 : 2^n \\ \quad f^{n+1}_{2i+1,2j} := \dfrac{f^n_{i+1,j} + f^n_{i,j}}{2} + \alpha h_n \big(p^n_{i+1,j} - p^n_{i,j}\big), \\ \quad p^{n+1}_{2i+1,2j} := (1-\beta)\dfrac{f^n_{i+1,j} - f^n_{i,j}}{h_n} + \beta \dfrac{p^n_{i+1,j} + p^n_{i,j}}{2}, \\ \quad q^{n+1}_{2i+1,2j} := \dfrac{q^n_{i+1,j} + q^n_{i,j}}{2}, \end{array}} \tag{2.2}$$

$$\boxed{\begin{array}{l} \text{for } i = 0 : 2^n, \quad \text{for } j = 0 : 2^n - 1 \\ \quad f^{n+1}_{2i,2j+1} := \dfrac{f^n_{i,j+1} + f^n_{i,j}}{2} + \alpha k_n \big(q^n_{i,j+1} - q^n_{i,j}\big), \\ \quad p^{n+1}_{2i,2j+1} := \dfrac{p^n_{i,j+1} + p^n_{i,j}}{2}, \\ \quad q^{n+1}_{2i,2j+1} := (1-\beta)\dfrac{f^n_{i,j+1} - f^n_{i,j}}{k_n} + \beta \dfrac{q^n_{i,j+1} + q^n_{i,j}}{2}, \end{array}} \tag{2.3}$$

$$
\begin{aligned}
&\text{for } i = 0 : 2^n - 1, \quad \text{for } j = 0 : 2^n - 1 \\
&\quad f^{n+1}_{2i+1,2j+1} := \frac{f^n_{i,j} + f^n_{i+1,j} + f^n_{i,j+1} + f^n_{i+1,j+1}}{4} \\
&\qquad + \alpha h_n \frac{p^n_{i+1,j} - p^n_{i,j} + p^n_{i+1,j+1} - p^n_{i,j+1}}{2} \\
&\qquad + \alpha k_n \frac{q^n_{i,j+1} - q^n_{i,j} + q^n_{i+1,j+1} - q^n_{i+1,j}}{2} \\
&\quad p^{n+1}_{2i+1,2j+1} := (1-\beta) \frac{f^n_{i+1,j} - f^n_{i,j} + f^n_{i+1,j+1} - f^n_{i,j+1}}{2h_n} \\
&\qquad + \beta \frac{p^n_{i,j} + p^n_{i,j+1} + p^n_{i+1,j} + p^n_{i+1,j+1}}{4} \\
&\qquad + \beta k_n \frac{q^n_{i+1,j+1} - q^n_{i+1,j} + q^n_{i,j} - q^n_{i,j+1}}{4h_n}, \\
&\quad q^{n+1}_{2i+1,2j+1} := (1-\beta) \frac{f^n_{i,j+1} - f^n_{i,j} + f^n_{i+1,j+1} - f^n_{i+1,j}}{2h_n} \\
&\qquad + \beta \frac{q^n_{i,j} + q^n_{i+1,j} + q^n_{i,j+1} + q^n_{i+1,j+1}}{4} \\
&\qquad + \beta k_n \frac{p^n_{i+1,j+1} - p^n_{i,j+1} + p^n_{i,j} - p^n_{i+1,j}}{4h_n}.
\end{aligned}
\tag{2.4}
$$

In (2.1) we simply redefine the functions at the points on $\mathcal{P}_n \times \mathcal{Q}_n$ as points on a subset of $\mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}$. These points are marked by gray squares in Figure 2.1. In (2.2), (2.3), and (2.4) we compute new values at the new points marked by black circles in Figure 2.1.

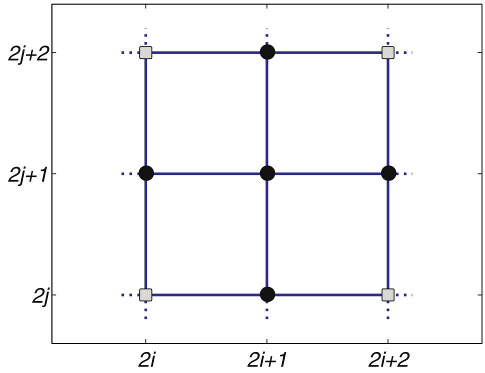

Figure 2.1: Recursive computation of f, p and q.

For $\alpha = -1/8, \beta = -1$ it was shown in [7] that we obtain the Sibson–Thomson interpolant on R proposed in [20]. In this case, the HRC^1-interpolant is a C^1 piecewise quadratic consisting of 16 individual pieces, see Figure 2.2. Moreover the cross boundary derivatives are linear functions along the outer boundary of $[a,b] \times [c,d]$.

It was shown in [7] that the HRC^1-algorithm is exact for bilinear functions for any value of α and β. It is exact for quadratic polynomials if and only if $\alpha = -1/8$ and exact for cubic polynomials if and only if $\alpha = -1/8$ and $\beta = -1/2$.

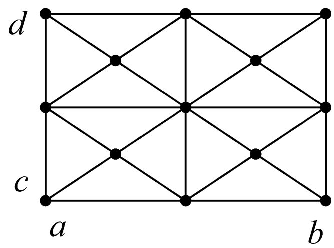

Figure 2.2: Sibson–Thomson subdivision of a rectangle.

We have simplified the construction of [7] and it depends on only two parameters α and β. This simplification gives new formulas for the computation of $f^{n+1}_{2i+1,2j+1}$, $p^{n+1}_{2i+1,2j+1}$ and $q^{n+1}_{2i+1,2j+1}$.

PROPOSITION 2.1.

(2.5)
$$
\begin{aligned}
&\textit{for } i = 0 : 2^n - 1, \quad \textit{for } j = 0 : 2^n - 1\\
&f^{n+1}_{2i+1,2j+1} = \frac{f^{n+1}_{2i+2,2j+1} + f^{n+1}_{2i,2j+1}}{2} + \alpha h_n \left(p^{n+1}_{2i+2,2j+1} - p^{n+1}_{2i,2j+1}\right),\\
&\qquad = \frac{f^{n+1}_{2i+1,2j+2} + f^{n+1}_{2i+1,2j}}{2} + \alpha k_n \left(q^{n+1}_{2i+1,2j+2} - q^{n+1}_{2i+1,2j}\right),\\
&p^{n+1}_{2i+1,2j+1} = (1-\beta)\frac{f^{n+1}_{2i+2,2j+1} - f^{n+1}_{2i,2j+1}}{h_n} + \beta\frac{p^{n+1}_{2i+2,2j+1} + p^{n+1}_{2i,2j+1}}{2}\\
&\qquad + \frac{k_n}{h_n}\big((1-\beta)\alpha - \beta/4\big)\\
&\qquad \times \left(q^{n+1}_{2i+2,2j} - q^{n+1}_{2i+2,2j+2} + q^{n+1}_{2i,2j+2} - q^{n+1}_{2i,2j}\right),\\
&q^{n+1}_{2i+1,2j+1} = (1-\beta)\frac{f^{n+1}_{2i+1,2j+2} - f^{n+1}_{2i+1,2j}}{k_n} + \beta\frac{q^{n+1}_{2i+1,2j+2} + q^{n+1}_{2i+1,2j}}{2}\\
&\qquad + \frac{h_n}{k_n}\big((1-\beta)\alpha - \beta/4\big)\\
&\qquad \times \left(p^{n+1}_{2i,2j+2} - p^{n+1}_{2i+2,2j+2} + p^{n+1}_{2i+2,2j} - p^{n+1}_{2i,2j}\right).
\end{aligned}
$$

PROOF. For $n \in \mathbb{N}$, $i,j \in \{0,\dots,2^n-1\}$, the first formula of (2.4) can be written:

$$f^{n+1}_{2i+1,2j+1} = \frac{1}{2}\left[\frac{f^n_{i+1,j+1}+f^n_{i+1,j}}{2} + \alpha k_n\big(q^n_{i+1,j+1}-q^n_{i+1,j}\big)\right.$$
$$\left. + \frac{f^n_{i,j+1}+f^n_{i,j}}{2} + \alpha k_n\big(q^n_{i,j+1}-q^n_{i,j}\big)\right]$$
$$+ \alpha h_n\left[\frac{p^n_{i+1,j+1}+p^n_{i+1,j}}{2} - \frac{p^n_{i,j+1}+p^n_{i,j}}{2}\right].$$

Using (2.3) this gives the first formula in (2.5). The proof of the second formula is similar.

We write the second formula of (2.4)

$$p^{n+1}_{2i+1,2j+1} = \frac{1-\beta}{h_n}\left[\frac{f^n_{i+1,j+1}+f^n_{i+1,j}}{2} + \alpha k_n\big(q^n_{i+1,j+1}-q^n_{i+1,j}\big)\right]$$
$$- \frac{1-\beta}{h_n}\left[\frac{f^n_{i,j+1}+f^n_{i,j}}{2} + \alpha k_n\big(q^n_{i,j+1}-q^n_{i,j}\big)\right]$$
$$+ \frac{\beta}{2}\left[\frac{p^n_{i+1,j+1}+p^n_{i+1,j}}{2} + \frac{p^n_{i,j+1}+p^n_{i,j}}{2}\right]$$
$$+ \frac{k_n}{h_n}\big[(1-\beta)\alpha - \beta/4\big]\big[q^n_{i+1,j} - q^n_{i+1,j+1} + q^n_{i,j+1} - q^n_{i,j}\big].$$

Thanks to (2.1) and (2.3), we obtain the result. The last formula is symmetrical from the previous one. □

3 C^1-convergence of the algorithm.

We say that the scheme is C^1*-convergent* if, for any initial data, the functions f, p, and q can be extended from $\mathcal{P}\times\mathcal{Q}$ to continuous functions on $[a,b]\times[c,d]$ with $p = f_x$ and $q = f_y$. We call f defined either on $\mathcal{P}\times\mathcal{Q}$ or on $[a,b]\times[c,d]$ the HRC^1*-interpolant* to the data.

For the study of C^1-convergence it is enough to consider the construction on the unit square $[0,1]^2$. To see this, let $h = b-a$, $k = d-c$ and let again (a,c) be the south-west vertex of the initial rectangle $[a,b]\times[c,d]$. On $[0,1]^2$, we define the initial data $g(u,v) := f(a+uh, c+vk)$, $g_x(u,v) := hf_x(a+uh,c+vk)$, $g_y(u,v) := kf_y(a+uh,c+vk)$, $(u,v) \in \{0,1\}^2$. The constructions of f or g by formulas (2.1), (2.2), (2.3), and (2.4) are equivalent and at each step, we obtain $g(u,v) = f(a+uh,c+vk)$, $g_x(u,v) = hf_x(a+uh,c+vk)$ and $g_y(u,v) = kf_y(a+uh,c+vk)$, $(u,v) \in \{0,1/2^n,\dots,\ell/2^n,\dots,1\}^2$. Thus the C^1-convergence of f on $[a,b]\times[c,d]$ is equivalent to the C^1-convergence of g on $[0,1]^2$.

So let us begin with data on the vertices of the unit square $[0,1]^2$. For $n \geq 0$ and $i,j = 0,1,\ldots,2^n-1$ we define vectors of differences $U_{ij}^n \in \mathbb{R}^{12}$ as follows:

$$U_{ij}^n := \begin{bmatrix} q_{i+1,j}^n - q_{i,j}^n \\ p_{i+1,j+1}^n - p_{i+1,j}^n \\ q_{i+1,j+1}^n - q_{i,j+1}^n \\ p_{i,j+1}^n - p_{i,j}^n \\ p_{i+1,j}^n - p_{i,j}^n \\ q_{i+1,j+1}^n - q_{i+1,j}^n \\ p_{i+1,j+1}^n - p_{i,j+1}^n \\ q_{i,j+1}^n - q_{i,j}^n \\ \dfrac{f_{i+1,j}^n - f_{i,j}^n}{h_n} - \dfrac{p_{i+1,j}^n + p_{i,j}^n}{2} \\ \dfrac{f_{i+1,j+1}^n - f_{i+1,j}^n}{h_n} - \dfrac{q_{i+1,j+1}^n + q_{i+1,j}^n}{2} \\ \dfrac{f_{i+1,j+1}^n - f_{i,j+1}^n}{h_n} - \dfrac{p_{i+1,j+1}^n + p_{i,j+1}^n}{2} \\ \dfrac{f_{i,j+1}^n - f_{i,j}^n}{h_n} - \dfrac{q_{i,j+1}^n + q_{i,j}^n}{2} \end{bmatrix}.$$

We then have

LEMMA 3.1. *Suppose we can find a vector norm* $\|\cdot\|$ *on* $\mathbb{R}^{12}$ *and positive constants* c,ρ *with* $\rho < 1$ *such that*

$$\left\|U_{ij}^n\right\| \leq c\rho^n, \quad \textit{for } i,j = 0,\ldots,2^n-1.$$

Then the HRC^1*-algorithm is* C^1*-convergent.*

PROOF. The proof of the C^1-convergence on $[0,1]^2$ is detailed in [7]. To summarize, with the hypothesis, it can be proved that p and q are uniformly continuous on the dyadic points so that they can be extended into continuous functions on $[0,1]^2$. Then we extend f and prove that $f_x = p$ and $f_y = q$ using the four last components of $U_{i,j}^n$. ☐

To bound the vectors U_{ij}^n we will use the following recurrence relations.

PROPOSITION 3.2. *We have*

$$U_{ij}^{n+1} = \Lambda^{(1)} U_{ij}^n, \quad U_{i+1,j}^{n+1} = \Lambda^{(2)} U_{ij}^n,$$
$$U_{i+1,j+1}^{n+1} = \Lambda^{(3)} U_{ij}^n, \quad U_{i,j+1}^{n+1} = \Lambda^{(4)} U_{ij}^n,$$

where $\Lambda^{(1)}$, $\Lambda^{(2)}$, $\Lambda^{(3)}$, $\Lambda^{(4)}$ *are 4 matrices in* $\mathbb{R}^{12\times 12}$ *depending only on the 2 parameters* α,β *of algorithm* HRC^1*. Explicit formulas for the matrices are as follows:*

$$\Lambda^{(i)} = \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} & \Lambda_{13}^{(i)} \\ 0 & \Lambda_{22}^{(i)} & \Lambda_{23}^{(i)} \\ 0 & \Lambda_{32}^{(i)} & \Lambda_{33}^{(i)} \end{bmatrix} = \begin{bmatrix} \Lambda_{11}^{(i)} & \ldots \\ 0 & M^{(i)} \end{bmatrix}$$

with $\Lambda_{jk}^{(i)} \in \mathbb{R}^{4\times 4}$ *and* $M^{(i)} \in \mathbb{R}^{8\times 8}$*. More specifically:*

$$\Lambda_{11}^{(1)} = \begin{bmatrix} \frac12 & 0 & 0 & 0 \\ 0 & \frac14 & 0 & \frac14 \\ \frac14 & 0 & \frac14 & 0 \\ 0 & 0 & 0 & \frac12 \end{bmatrix}, \qquad \Lambda_{11}^{(2)} = \begin{bmatrix} \frac12 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & 0 \\ \frac14 & 0 & \frac14 & 0 \\ 0 & \frac14 & 0 & \frac14 \end{bmatrix},$$

$$\Lambda_{11}^{(3)} = \begin{bmatrix} \frac14 & 0 & \frac14 & 0 \\ 0 & \frac12 & 0 & 0 \\ 0 & 0 & \frac12 & 0 \\ 0 & \frac14 & 0 & \frac14 \end{bmatrix}, \qquad \Lambda_{11}^{(4)} = \begin{bmatrix} \frac14 & 0 & \frac14 & 0 \\ 0 & \frac14 & 0 & \frac14 \\ 0 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac12 \end{bmatrix},$$

$$\left(\Lambda_{12}^{(1)}\Lambda_{13}^{(1)}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta/4 & 0 & -\beta/4 & \frac{\beta-1}{2} & 0 & \frac{1-\beta}{2} & 0 \\ -\beta/4 & 0 & \beta/4 & 0 & 0 & \frac{1-\beta}{2} & 0 & \frac{\beta-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$M^{(1)} = \begin{bmatrix} \frac12 & 0 & 0 & 0 & 1-\beta & 0 & 0 & 0 \\ -\frac{\beta}{4} & \frac14 & \frac{\beta}{4} & \frac14 & 0 & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} \\ \frac14 & \frac{\beta}{4} & \frac14 & -\frac{\beta}{4} & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} & 0 \\ 0 & 0 & 0 & \frac12 & 0 & 0 & 0 & 1-\beta \\ \frac14+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ +\frac{\beta}{8}-\alpha & \frac18+\alpha & \alpha-\frac{\beta}{8} & \frac18+\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \\ \frac18+\alpha & \alpha-\frac{\beta}{8} & \frac18+\alpha & \frac{\beta}{8}-\alpha & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ 0 & 0 & 0 & \frac14+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{bmatrix}$$

$$M^{(2)} = \begin{bmatrix} \frac12 & 0 & 0 & 0 & \beta-1 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & 0 & 0 & 1-\beta & 0 & 0 \\ \frac14 & -\frac{\beta}{4} & \frac14 & \frac{\beta}{4} & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} & 0 \\ -\frac{\beta}{4} & \frac14 & \frac{\beta}{4} & \frac14 & 0 & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} \\ -\frac14-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ 0 & \frac14+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ -\frac18-\alpha & -\frac{\beta}{8}+\alpha & -\frac18-\alpha & -\alpha+\frac{\beta}{8} & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ \frac{\beta}{8}-\alpha & \frac18+\alpha & \alpha-\frac{\beta}{8} & \frac18+\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \end{bmatrix}$$

$$M^{(3)} = \begin{bmatrix} \frac14 & -\frac{\beta}{4} & \frac14 & \frac{\beta}{4} & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} & 0 \\ 0 & \frac12 & 0 & 0 & 0 & \beta-1 & 0 & 0 \\ 0 & 0 & \frac12 & 0 & 0 & 0 & \beta-1 & 0 \\ \frac{\beta}{4} & \frac14 & -\frac{\beta}{4} & \frac14 & 0 & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} \\ -\frac18-\alpha & -\frac{\beta}{8}+\alpha & -\frac18-\alpha & -\alpha+\frac{\beta}{8} & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ 0 & -\frac14-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ 0 & 0 & -\frac14-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ -\alpha+\frac{\beta}{8} & -\frac18-\alpha & -\frac{\beta}{8}+\alpha & -\frac18-\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \end{bmatrix}$$

$$M^{(4)} = \begin{bmatrix} \frac{1}{4} & \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} & 0 \\ \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1}{4} & 0 & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 \\ \frac{1}{8}+\alpha & \alpha-\frac{\beta}{8} & \frac{1}{8}+\alpha & \frac{\beta}{8}-\alpha & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ -\alpha+\frac{\beta}{8} & -\frac{1}{8}-\alpha & -\frac{\beta}{8}+\alpha & -\frac{1}{8}-\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \\ 0 & 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{bmatrix}.$$

PROOF. These relations were shown in [7] using a a computer algebra system. □

We will not need the explicit form of the matrices $[\Lambda_{12}^{(i)}\Lambda_{13}^{(i)}]$ for $i \in \{1,2,3,4\}$. One can obtain all of them from $[\Lambda_{12}^{(1)}\Lambda_{13}^{(1)}]$ by permutations of rows and columns.

We mention that in [7] it was proved that the scheme is C^1-convergent if and only if the generalized spectral radius of $\Sigma = \{\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}\}$ satisfies $\hat{\rho}(\Sigma) < 1$. The following analysis is maybe somewhat simpler. We start with a proposition.

PROPOSITION 3.3. *If there exists a vector norm $\|\cdot\|$ on $\mathbb{R}^{12}$ and a number $\rho < 1$ such that the associated matrix operator norm satisfies $\|\Lambda^{(i)}\| \le \rho$ for $i = 1,2,3,4$ then the scheme is C^1-convergent. Moreover the functions p and q are Hölder continuous with exponent $-log_2(\rho)$.*

PROOF. It is enough to prove the proposition on the square $[0,1]^2$. That the scheme is C^1-convergent follows immediately from Lemma 3.1.

The proof that p and q are Hölder continuous is similar to a proof in dimension one in [17]. In the following proof we will use the function notation for the sequences $\{U_{ij}^n\}_{i,j}, \{f_{ij}^n\}_{i,j}, \{p_{ij}^n\}_{i,j}$, and $\{q_{ij}^n\}_{i,j}$. Thus if $x := i2^{-n}$ and $y := j2^{-n}$ then we write U_{ij}^n and p_{ij}^n as $U^n(x,y)$ and $p^n(x,y)$. We recall that $\mathcal{P}_\ell = \{k2^{-\ell}, k = 0, \dots, 2^\ell\}, \ell \in \mathbb{N}$ is the set of dyadic points at step ℓ on $[0,1]$ and we write $h_\ell = 1/2^\ell$. With the hypothesis, for $(x,y) \in {\mathcal{P}_\ell}^2, x \ne 1, y \ne 1$, we have $\|U^\ell(x,y)\| \le \rho^\ell \|U^0(0,0)\|$ for $\ell \ge 0$. Using the equivalence of the norms in $\mathbb{R}^{12}$, this implies that $\|U^\ell(x,y)\|_\infty \le c_2\rho^\ell$ for some positive constant c_2 independent of U^ℓ. In particular, this holds for components 4 and 5 of $U^\ell(x,y)$ and we deduce that

$$\begin{aligned} &|p(x \pm h_\ell, y) - p(x,y)| \le c_2\rho^\ell, \quad \text{with } (x,y), (x \pm h_\ell, y) \in {\mathcal{P}_\ell}^2 \\ &|p(x, y \pm h_\ell) - p(x,y)| \le c_2\rho^\ell, \quad \text{with } (x,y), (x, y \pm h_\ell) \in {\mathcal{P}_\ell}^2. \end{aligned} \tag{3.1}$$

Suppose that $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are 2 points in $[0,1]^2$. Let n be the unique nonnegative integer such that $2^{-n-1} < \|P_1 - P_2\|_\infty \le 2^{-n}$. Then $|x_2 - x_1| \le 2^{-n}$ and $|y_2 - y_1| \le 2^{-n}$ and there exist $x, y \in \mathcal{P}_n$ such that $|x_j - x| \le 2^{-n}$ and $|y_j - y| \le 2^{-n}$ for $j = 1,2$. Thus $P := (x,y) \in {\mathcal{P}_n}^2$ is such that $\|P_j - P\|_\infty \le 2^{-n}$ for $j = 1,2$.

To prove that $|p(P_1) - p(P)| \le c_3\rho^n$ for some constant c_3 we write $P_1 = P + \sum_{i=1}^\infty (u_i, v_i)2^{-i-n}$ with u_i and v_i in $\{0,1,-1\}$. We define the sequence

$\{\hat{P}_j\} := \{(\hat{x}_j, \hat{y}_j)\}$ by $\hat{P}_0 = P$ and $\hat{P}_j = \hat{P}_{j-1} + (u_j, v_j)2^{-j-n}$, for $j \geq 1$. Then $\hat{P}_j \in {\mathcal{P}_{n+j}}^2$ and

$$|p(\hat{P}_j) - p(\hat{P}_{j-1})| \leq |p(\hat{x}_j, \hat{y}_j) - p(\hat{x}_j, \hat{y}_{j-1})| + |p(\hat{x}_j, \hat{y}_{j-1}) - p(\hat{x}_{j-1}, y_{j-1})|.$$

Since $(\hat{x}_j, \hat{y}_j), (\hat{x}_j, \hat{y}_{j-1})$ and $(\hat{x}_{j-1}, \hat{y}_{j-1})$ are in ${\mathcal{P}_{n+j}}^2$ we can bound them using (3.1) with $\ell = n + j$ and we obtain $|p(\hat{P}_j) - p(\hat{P}_{j-1})| \leq 2c_2\rho^{n+j}$ so that $|p(P_1) - p(P)| \leq \sum_{j=1}^{\infty} 2c_2\rho^{n+j} = \frac{2c_2}{1-\rho}\rho^{n+1}$.

With the same upper bound for $|p(P_2) - p(P)|$, we deduce that $|p(P_2) - p(P_1)| \leq c_4\rho^{n+1}$ with $c_4 = \frac{4c_2}{1-\rho}$.

To conclude, notice that since $\|P_1 - P_2\|_\infty > 2^{-n-1}$ then

$$|p(P_2) - p(P_1)| \leq c_4\rho^{n+1} = c_4 2^{(-n-1)(-\log_2 \rho)} < c_4\|P_1 - P_2\|_\infty^{-\log_2(\rho)}.$$

A similar inequality holds for the function q. □

To find a good norm on $\mathbb{R}^{12}$, we use the following well known result:

LEMMA 3.4. *Corresponding to a positive integer d, a nonsingular matrix $P \in \mathbb{R}^{d\times d}$ and a vector norm $\|\cdot\|$ on $\mathbb{R}^d$ we define a vector norm on $\mathbb{R}^d$ by $\|V\|_1 := \|P^{-1}V\|$. Then the associated matrix operator norm $\|\cdot\|$ is given by $\|A\|_1 = \|P^{-1}AP\|$ for any matrix $A \in \mathbb{R}^{d\times d}$.*

PROOF. Clearly $\|\cdot\|_1$ defines a norm on $\mathbb{R}^d$. Now if $A \in \mathbb{R}^{d\times d}$ then

$$\|A\|_1 := \max_{V\neq 0} \frac{\|AV\|_1}{\|V\|_1} = \max_{V\neq 0} \frac{\|P^{-1}AV\|}{\|P^{-1}V\|} = \max_{U\neq 0} \frac{\|P^{-1}APU\|}{\|U\|} = \|P^{-1}AP\|.$$

□

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two vector norms on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. For a matrix $A \in \mathbb{R}^{d_1\times d_2}$ we write $\|A\|_{12}$ for the associated mixed matrix operator norm $\|A\|_{12} := \max_{V\in\mathbb{R}^{d_2}, V\neq 0} \frac{\|AV\|_1}{\|V\|_2}$.

LEMMA 3.5. *Suppose for positive integers d_1 and d_2 that Σ is a set of square matrices $\{A\}$ of order $d := d_1 + d_2$ that are written by blocks as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.2}$$

with diagonal blocks $A_{ii} \in \mathbb{R}^{d_i\times d_i}$ for $i = 1, 2$. For $i = 1, 2$, let $\|\cdot\|_i$ be two vector norms on $\mathbb{R}^{d_i}$ and for $i, j = 1, 2$, let γ_{ij} be positive constants such that for any $A \in \Sigma$ the estimates $\|A_{ij}\|_{ij} \leq \gamma_{ij}$ hold. If

$$\gamma_{11} < 1, \quad \gamma_{22} < 1, \quad \textit{and} \quad \gamma_{21}\gamma_{12} < (1 - \gamma_{11})(1 - \gamma_{22})$$

then we can find a matrix norm on $\mathbb{R}^{d\times d}$ such that any $A \in \Sigma$ has norm less than 1.

PROOF. On $\mathbb{R}^d$, we define a norm $\|\cdot\|_\theta$ depending on a parameter $\theta > 0$. If $V = (X, Y)^T$ with $X \in \mathbb{R}^{d_1}$ and $Y \in \mathbb{R}^{d_2}$ then $\|V\|_\theta := \|X\|_1 + \theta\|Y\|_2$.

Then for any matrix $A \in \mathbb{R}^{d\times d}$, we have:

$$\begin{aligned}\|AV\|_\theta &= \|A_{11}X + A_{12}Y\|_1 + \theta\|A_{21}X + A_{22}Y\|_2\\ &\le \|A_{11}\|_{11}\|X\|_1 + \|A_{12}\|_{12}\|Y\|_2 + \theta\|A_{21}\|_{21}\|X\|_1 + \theta\|A_{22}\|_{22}\|Y\|_2\\ &= (\|A_{11}\|_{11} + \theta\|A_{21}\|_{21})\|X\|_1 + (\|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})(\theta\|Y\|_2)\\ &\le \max(\|A_{11}\|_{11} + \theta\|A_{21}\|_{21}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})\|V\|_\theta.\end{aligned}$$

We deduce that

$$\|A\|_\theta \le \max(\|A_{11}\|_{11} + \theta\|A_{21}\|_{21}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22}), \quad A \in \mathbb{R}^{d\times d}. \tag{3.3}$$

$\|A\|_\theta < 1$, as soon as $\|A_{11}\|_{11} + \theta\|A_{21}\|_{21} < 1$ and $\|A_{12}\|_{12}/\theta + \|A_{22}\|_{22} < 1$. Since, for any $A \in \Sigma$, $\|A_{ij}\| \le \gamma_{ij}$, $i, j = 1, 2$, it suffices that $\gamma_{11} + \theta\gamma_{21} < 1$ and $\gamma_{12}/\theta + \gamma_{22} < 1$. If $\gamma_{11} < 1$ and $\gamma_{22} < 1$, these conditions are satisfied whenever there exists a real number $\theta > 0$ such that $\frac{\gamma_{12}}{1-\gamma_{22}} < \theta < \frac{1-\gamma_{11}}{\gamma_{21}}$. Since $\gamma_{21}\gamma_{12} < (1-\gamma_{11})(1-\gamma_{22})$ we can find such a θ. □

LEMMA 3.6. *Suppose in Lemma 3.5 that Σ is a finite family of matrices of the form* (3.2) *with $A_{21} = 0$. If there exists a real number $c > 0$ such that for any $A \in \Sigma$, $\|A_{11}\|_{11} \le c$ and $\|A_{22}\|_{22} < c$ then there exists a matrix norm such that for all $A \in \Sigma$, we have $\|A\| \le c$.*

PROOF. Using (3.3) in the previous lemma, $\|A\|_\theta \le \max(\|A_{11}\|_{11}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})$. Now $\|A_{11} \le c$ and $\|A_{22}\|_{22} < c$. Since the set Σ is finite, we can find a real number $\theta > 0$ such that for any $A \in \Sigma$, $\|A_{12}\|_{12}/\theta$ is small enough to get $\|A\|_\theta \le c$. □

Now we have the tools to study the C^1-convergence of the algorithm.

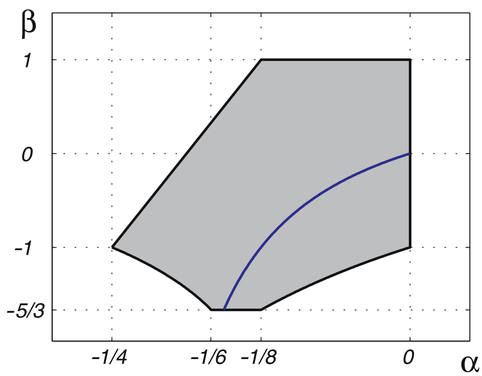

Figure 3.1: The region $\mathcal{R}$ in Theorem 3.7 together with the curve $\alpha = \beta/(4(1-\beta))$.

THEOREM 3.7. *The algorithm HRC^1 is C^1 convergent if (α, β) belongs to the region*

$$\mathcal{R} := \left\{(\alpha, \beta) : -\tfrac{1}{4} < \alpha < 0 \quad \textit{and} \quad l(\alpha) < \beta < u(\alpha)\right\}, \tag{3.4}$$

where

$$l(\alpha) := \begin{cases} 8\alpha - 2 + \sqrt{(8\alpha+1)(8\alpha-7)} & \text{if } -\frac{1}{4} < \alpha < -\frac{1}{6}, \\ -\frac{5}{3} & \text{if } -\frac{1}{6} \le \alpha < -\frac{1}{8}, \\ \frac{2\alpha-1}{2\alpha+1} & \text{if } -\frac{1}{8} \le \alpha < 0, \end{cases} \tag{3.5}$$

$$u(\alpha) := \begin{cases} 16\alpha + 3, & \text{if } -\frac{1}{4} < \alpha < -\frac{1}{8}, \\ 1 & \text{if } -\frac{1}{8} \le \alpha < 0. \end{cases} \tag{3.6}$$

PROOF. Let $\|A\|_\infty = \max_{i=1,\dots,d}(\sum_{j=1}^d |a_{ij}|)$ be the matrix norm on $\mathbb{R}^{d\times d}$ associated with the vector norm $\|V\|_\infty = \max_{k=1,\dots,d}(|v_k|)$ on $\mathbb{R}^d$. Since $\|\Lambda_{11}^{(\ell)}\|_\infty = 1/2$, using Lemma 3.6 we get a matrix norm such that $\|\Lambda^{(\ell)}\| < 1$ as soon as there exists a matrix norm such that $\|M^{(\ell)}\| < 1$, $\ell = 1, \dots, 4$.

Let $P_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}$. We compute $N^{(\ell)} := P^{-1}M^{(\ell)}P = \begin{bmatrix} N_{11}^{(\ell)} & N_{12}^{(\ell)} \\ N_{21}^{(\ell)} & N_{22}^{(\ell)} \end{bmatrix}$ for $\ell = 1, \dots, 4$. By Lemma 3.4 we know that it suffices to find a matrix norm such that $\|N^{(\ell)}\| < 1$, for $\ell = 1, 2, 3, 4$. The computation gives:

$$N_{11}^{(1)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 1 & \beta \\ 0 & 2 & -\beta & -1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & -\beta & 1 \end{bmatrix} \qquad N_{12}^{(1)} = \frac{1-\beta}{2}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{21}^{(1)} = \begin{bmatrix} 1/4+2\alpha & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & 1/4+2\alpha & -\alpha+\beta/8 & -1/8-\alpha \\ 0 & 0 & 1/8+\alpha & -\alpha+\beta/8 \\ 0 & 0 & -\alpha+\beta/8 & 1/8+\alpha \end{bmatrix} \qquad N_{22}^{(1)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(2)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 1 & -\beta \\ 0 & 2 & -\beta & 1 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix} \qquad N_{12}^{(2)} = \frac{1-\beta}{2}\begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{21}^{(2)} = \begin{bmatrix} -1/4-2\alpha & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & 1/4+2\alpha & -\alpha+\beta/8 & 1/8+\alpha \\ 0 & 0 & -1/8-\alpha & -\alpha+\beta/8 \\ 0 & 0 & \alpha-\beta/8 & 1/8+\alpha \end{bmatrix} \qquad N_{22}^{(2)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(3)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & -1 & -\beta \\ 0 & 2 & \beta & 1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & -\beta & 1 \end{bmatrix} \qquad N_{12}^{(3)} = \frac{1-\beta}{2}\begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$N_{21}^{(3)} = \begin{bmatrix} -1/4-2\alpha & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & -1/4-2\alpha & -\alpha+\beta/8 & -1/8-\alpha \\ 0 & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & 0 & \alpha-\beta/8 & -1/8-\alpha \end{bmatrix} \quad N_{22}^{(3)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(4)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & -1 & \beta \\ 0 & 2 & \beta & -1 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix} \qquad N_{12}^{(4)} = \frac{1-\beta}{2}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$N_{21}^{(4)} = \begin{bmatrix} 1/4+2\alpha & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & -1/4-2\alpha & -\alpha+\beta/8 & 1/8+\alpha \\ 0 & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & 0 & -\alpha+\beta/8 & -1/8-\alpha \end{bmatrix} \quad N_{22}^{(4)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In $\mathbb{R}^4$, we use the norm $\|\cdot\|_\theta, \theta > 0$ defined at the beginning of the proof of Lemma 3.5, using $\|\cdot\|_\infty$ in $\mathbb{R}^4$, i.e. $\|U\|_\theta = \|X\|_\infty + \theta\|Y\|_\infty$ where $U = \begin{bmatrix} X \\ Y \end{bmatrix}$, $X, Y \in \mathbb{R}^2$. Using (3.3), we deduce that for $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \in \mathbb{R}^{4\times 4}$ with $A_{ij} \in \mathbb{R}^{2\times 2}$, we have $\|A\|_\theta \le \max(\|A_{11}\|_\infty, \|A_{12}\|_\infty/\theta + \|A_{22}\|_\infty)$.

For $\ell = 1, 2, 3, 4$, we can then bound $\|N_{ij}^{(\ell)}\|_\theta$. Let $\mu := 1 + \frac{1}{\theta} > 1$ and assume $\mu < 2$. Then

$$\begin{aligned} \|N_{11}^{(\ell)}\|_\theta &\le \tfrac{1}{4}\max(2, \mu(1-\beta)) =: \gamma_{11}, \\ \|N_{12}^{(\ell)}\|_\theta &\le |1-\beta| =: \gamma_{12}, \\ \|N_{21}^{(\ell)}\|_\theta &\le \max(|1/4+2\alpha|, \mu(|-\alpha+\beta/8| + |1/8+\alpha|)) =: \gamma_{21}, \\ \|N_{22}^{(\ell)}\|_\theta &\le \frac{|1+\beta|}{2} =: \gamma_{22}. \end{aligned}$$

We need to bound the $\gamma's$. The analysis below shows that it is enough to consider (α, β) in the rectangle $[-\frac{1}{4}, 0] \times [-2, 1]$. To compute γ_{21}, which is the most difficult, we divide the rectangle $[-\frac{1}{4}, 0] \times [-2, 1]$ into open subsets $R_1, \dots, R_6$ as shown in Figure 3.2. In these regions a lengthy, but straightforward calculation gives the following values for the numbers γ_{ij} and the quantity $\pi_\gamma := (1-\gamma_{11})(1-\gamma_{22})$:

	R_1	R_2	R_3	R_4	R_5	R_6
γ_{11}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\mu}{4}(1-\beta)$	$\frac{\mu}{4}(1-\beta)$	$\frac{\mu}{4}(1-\beta)$
γ_{22}	$\frac{1}{2}(1+\beta)$	$\frac{1}{2}(1+\beta)$	$\frac{1}{2}(1+\beta)$	$-\frac{1}{2}(1+\beta)$	$-\frac{1}{2}(1+\beta)$	$-\frac{1}{2}(1+\beta)$
γ_{12}	$1-\beta$	$1-\beta$	$1-\beta$	$1-\beta$	$1-\beta$	$1-\beta$
γ_{21}	$-\mu(2\alpha+\frac{1-\beta}{8})$	$\frac{\mu}{8}(1+\beta)$	$2\alpha+\frac{1}{4}$	$-2\alpha-\frac{1}{4}$	$-\frac{\mu}{8}(1+\beta)$	$\mu(2\alpha+\frac{1-\beta}{8})$
π_γ	$\frac{1}{4}(1-\beta)$	$\frac{1}{4}(1-\beta)$	$\frac{1}{4}(1-\beta)$	ν	ν	ν

where $\nu := \frac{1}{8}(3+\beta)^2 - \epsilon$, and where $\epsilon > 0$ can be made arbitrary small by choosing θ sufficiently big.

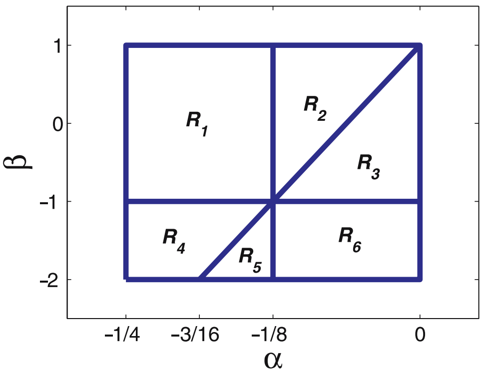

Figure 3.2: The subsets $R_1, \dots, R_6$ used in the proof of Theorem 3.7.

We need to compute subsets S_j of R_j so that $\gamma_{12}\gamma_{21} < \pi_\gamma$ for $(\alpha, \beta) \in S_j$, $j = 1, \dots, 6$.

- On R_1, we need $-\mu(1-\beta)(2\alpha+\frac{1-\beta}{8}) < \frac{1-\beta}{4}$. This is satisfied if $\beta < 16\alpha+3$.
- On R_2, the condition is $\mu(1-\beta)\frac{1+\beta}{8} < \frac{1-\beta}{4}$ which holds if $\beta < 1$.
- On R_3, we should have $(1-\beta)(\frac{1}{4}+2\alpha) < \frac{1-\beta}{4}$. This is true for $\alpha < 0$.
- On R_4, the inequality is $(1-\beta)(-\frac{1}{4}-2\alpha) < \frac{(3+\beta)^2}{8} - \epsilon$. Since this should hold for all $\epsilon > 0$ we can drop the ϵ (this is also true for R_5, and R_6) and we obtain

$$\alpha < \frac{11+4\beta+\beta^2}{16(1-\beta)} \quad \text{or} \quad \beta < 8\alpha - 2 + \sqrt{(8\alpha+1)(8\alpha-7)}.$$

- The condition on R_5 takes the form $(1-\beta)\frac{1+\beta}{8} < \frac{(3+\beta)^2}{8}$ which holds if $\beta > -\frac{5}{3}$.
- Finally on R_6, the inequality $\mu(1-\beta)(2\alpha+\frac{1-\beta}{8}) < \frac{(3+\beta)^2}{8}$ is true for $\beta < \frac{2\alpha-1}{2\alpha+1}$.

This defines the subregions S_j of R_j for $j = 1, \dots, 6$. It remains to show that the result also holds on the curve segments forming the interior boundaries between the regions. These curves can be identified as $\beta = -1$, $\alpha = -1/8$, and $\beta = 16\alpha + 1$. We divide these curves into segments as follows:

$$C_{11} := \{(\alpha,\beta) : -\tfrac{1}{4} < \alpha < \tfrac{1}{8},\ \beta = -1\}, \qquad C_{12} := \{(\alpha,\beta) : -\tfrac{1}{8} \le \alpha < 0,\ \beta = -1\},$$
$$C_{21} := \{(\alpha,\beta) : \alpha = -\tfrac{1}{8},\ -\tfrac{5}{3} < \beta < -1\}, \qquad C_{22} := \{(\alpha,\beta) : \alpha = -\tfrac{1}{8},\ -1 \le \beta < \psi\},$$
$$C_{23} := \{(\alpha,\beta) : \alpha = -\tfrac{1}{8},\ \psi \le \beta < 1\}, \qquad C_{31} := \{(\alpha, 16\alpha+1) : -\tfrac{1}{6} < \alpha < -\tfrac{1}{8}\},$$
$$C_{32} := \{(\alpha, 16\alpha+1) : -\tfrac{1}{8} \le \alpha < -\tfrac{1}{8\mu}\}, \qquad C_{33} := \{(\alpha, 16\alpha+1) : -\tfrac{1}{8\mu} \le \alpha < 0\},$$

where $\psi := -1 + \frac{2}{1+\theta}$. The values of γ_{ij} and $\delta := (1-\gamma_{11})(1-\gamma_{22}) - \gamma_{12}\gamma_{21}$ on the different segments are shown in the following table:

Segment	γ_{11}	γ_{22}	γ_{12}	γ_{21}	δ
C_{11}	$\mu/2$	0	2	$-2\mu(\alpha+\frac{1}{8})$	$1+4\mu\alpha$
C_{12}	$\mu/2$	0	2	$2\mu(\alpha+\frac{1}{8})$	$1-\mu-4\mu\alpha$
C_{21}	$\frac{\mu}{4}(1-\beta)$	$-\frac{1+\beta}{2}$	$1-\beta$	$-\frac{\mu}{8}(\beta+1)$	$\frac{1}{4}(6-\mu+\beta(\mu+2))$
C_{22}	$\frac{\mu}{4}(1-\beta)$	$\frac{1+\beta}{2}$	$1-\beta$	$\frac{\mu}{8}(\beta+1)$	$\frac{1}{4}(1-\beta)(2-\mu)$
C_{23}	$\frac{1}{2}$	$\frac{1+\beta}{2}$	$1-\beta$	$\frac{\mu}{8}(\beta+1)$	$\frac{1}{8}(1-\beta)(2-\mu-\mu\beta)$
C_{31}	$-4\mu\alpha$	$-1-8\alpha$	-16α	$-\frac{\mu}{4}(1+8\alpha)$	$2+4\alpha(2+\mu)$
C_{32}	$-4\mu\alpha$	$1+8\alpha$	-16α	$\frac{\mu}{4}(1+8\alpha)$	$-4\alpha(2-\mu)$
C_{33}	$\frac{1}{2}$	$1+8\alpha$	-16α	$\frac{\mu}{4}(1+8\alpha)$	$4(\mu-1)\alpha+32\mu\alpha^2$

where as before we set $\mu := \frac{1}{\theta}+1$.

We have C^1-convergence for a specific value of (α,β) provided we can find a $\theta > 0$ so that $\delta > 0$. This is always possible for any point in the open interval (see Figure 3.3). □

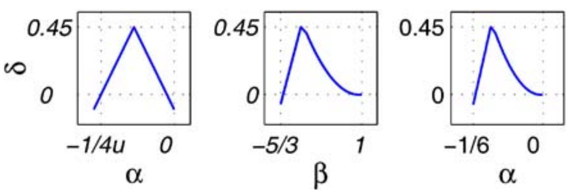

Figure 3.3: The value of $\delta = (1-\gamma_{11})(1-\gamma_{22}) - \gamma_{12}\gamma_{21}$ on the curve segments defined by $\beta = -1$ (left), $\alpha = -\frac{1}{8}$ (center) and $\beta = 16\alpha + 1$ (right) corresponding to $\theta = 10$ or $\mu = 1.1$.

COROLLARY 3.8. *For* $\alpha = \frac{\beta}{4(1-\beta)}$ *and* $\beta \in [-5/3, 0)$*, the scheme* HRC^1 *is* C^1*-convergent.*

PROOF. If $\beta \in [-5/3, 0)$ and $\alpha = \frac{\beta}{4(1-\beta)}$, then $(\alpha,\beta) \in \mathcal{R}$, see Figure 3.1. □

4 The control grid.

In order to obtain a geometric formulation of the HRC^1-algorithm we define *control coefficients* a_{ij} and *control points* A_{ij} relative to the rectangle $R = [a,b] \times [c,d]$ as follows:

$$A_{00} = (a, c, a_{00}), \qquad \text{where } a_{00} = f(a,c),$$

$$A_{10} = \left(a + \tfrac{h}{\lambda}, c, a_{10}\right), \qquad \text{where } a_{10} = f(a,c) + \tfrac{hp(a,c)}{\lambda},$$

$$A_{20} = \left(b - \tfrac{h}{\lambda}, c, a_{20}\right), \qquad \text{where } a_{20} = f(b,c) - \tfrac{hp(b,c)}{\lambda}, \tag{4.1}$$

$$
\begin{aligned}
A_{30} &= (b, c, a_{30}), && \text{where } a_{30} = f(b,c),\\
A_{31} &= \big(b, c + \tfrac{k}{\lambda}, a_{31}\big), && \text{where } a_{31} = f(b,c) + \tfrac{kq(b,c)}{\lambda},\\
A_{32} &= \big(b, d - \tfrac{k}{\lambda}, a_{32}\big), && \text{where } a_{32} = f(b,d) - \tfrac{kq(b,d)}{\lambda},\\
A_{33} &= (b, d, a_{33}), && \text{where } a_{33} = f(b,d),\\
A_{23} &= \big(b - \tfrac{h}{\lambda}, d, a_{23}\big), && \text{where } a_{23} = f(b,d) - \tfrac{hp(b,d)}{\lambda},\\
A_{13} &= \big(a + \tfrac{h}{\lambda}, d, a_{13}\big), && \text{where } a_{13} = f(a,d) + \tfrac{hp(a,d)}{\lambda},\\
A_{03} &= \big(a, d, a_{03}\big), && \text{where } a_{03} = f(a,d),\\
A_{02} &= \big(a, d - \tfrac{k}{\lambda}, a_{02}\big), && \text{where } a_{02} = f(a,d) - \tfrac{kq(a,c)}{\lambda},\\
A_{01} &= \big(a, c + \tfrac{k}{\lambda}, a_{01}\big), && \text{where } a_{01} = f(a,c) + \tfrac{kq(a,c)}{\lambda}.
\end{aligned}
$$

Here $h := b - a$, $k := d - c$ and $\lambda \geq 2$ is a real number to be chosen. The 12 control points are located on the boundary of R. We can obtain a control polygon-like structure by adding the four interiour points $A_{11} = A_{10} + A_{01} - A_{00}$, $A_{21} = A_{20} + A_{31} - A_{30}$, $A_{22} = A_{23} + A_{32} - A_{33}$, and $A_{12} = A_{13} + A_{02} - A_{03}$, see Figure 4.1.

If f is the HRC^1-interpolant constructed from the given data at the vertices of R then the parametric surface $(x, y, f(x, y))$ with $(x, y) \in R$ interpolates the corner control points $A_{00}, A_{03}, A_{30}, A_{33}$. Moreover each corner rectangle in the control polygon defines a plane which is part of the tangent plane at that vertex. For example the plane containing the four points A_{00}, A_{10}, A_{01}, and A_{11} defines the tangent plane to the surface at A_{00}.

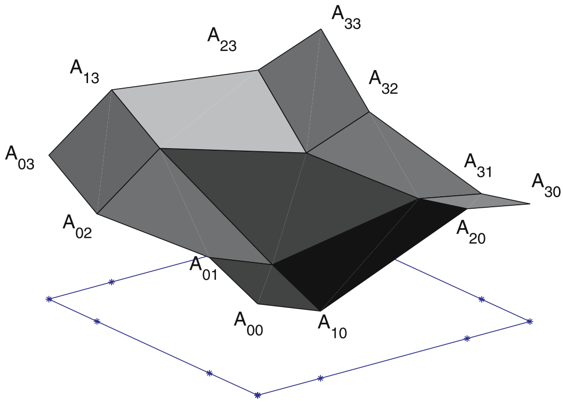

Figure 4.1: Control grid.

After one step of subdivision the rectangle R is divided into four subrectangles (cf. Figure 2.1). On each of the four sub-rectangles we can compute new control points $\bar{A}_{ij}$. To compute these control points we can use (4.1) shifted to each subrectangle. In particular, we replace h and k by $h/2$ and $k/2$ respectively.

By using (2.1), (2.2), (2.3), and (2.4) and inverting the formulas in (4.1) it is possible to express the new control coefficients $\{\bar{a}_{ij}\}$ in terms of the original control coefficients a_{ij}. We restrict attention to the one parameter family given by $\alpha = \beta/(4(1-\beta))$. We also write $\lambda = u(1-\beta)$, where u is a free parameter to be chosen later. With the aid of a computer algebra system it can be proved that:

PROPOSITION 4.1. *Suppose* $\alpha = \beta/(4(1-\beta))$ *and* $\lambda = u(1-\beta)$. *With the indexing used in Figures 4.2 and 4.3 we have*

$$
\begin{bmatrix} \bar{a}_{0,0} \\ \bar{a}_{1,0} \\ \bar{a}_{2,0} \\ \bar{a}_{3,0} \\ \bar{a}_{4,0} \\ \bar{a}_{5,0} \\ \bar{a}_{6,0} \\ \bar{a}_{0,1} \\ \bar{a}_{3,1} \\ \bar{a}_{6,1} \\ \bar{a}_{0,2} \\ \bar{a}_{3,2} \\ \bar{a}_{6,2} \\ \bar{a}_{0,3} \\ \bar{a}_{1,3} \\ \bar{a}_{2,3} \\ \bar{a}_{3,3} \\ \bar{a}_{4,3} \\ \bar{a}_{5,3} \\ \bar{a}_{6,3} \\ \bar{a}_{0,4} \\ \bar{a}_{3,4} \\ \bar{a}_{6,4} \\ \bar{a}_{0,5} \\ \bar{a}_{3,5} \\ \bar{a}_{6,5} \\ \bar{a}_{0,6} \\ \bar{a}_{1,6} \\ \bar{a}_{2,6} \\ \bar{a}_{3,6} \\ \bar{a}_{4,6} \\ \bar{a}_{5,6} \\ \bar{a}_{6,6} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{vy}{4u} & \frac{1}{4}v\gamma & \frac{1}{4}w\gamma & \frac{wy}{4u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4}y & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{wy}{4u} & \frac{1}{4}w\gamma & \frac{1}{4}v\gamma & \frac{vy}{4u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4}x & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}x & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{vy}{4u} & 0 & 0 & 0 & \frac{1}{4}v\gamma & 0 & \frac{1}{4}w\gamma & 0 & \frac{wy}{4u} & 0 & 0 & 0 \\
\frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} \\
0 & 0 & 0 & \frac{vy}{4u} & 0 & \frac{1}{4}v\gamma & 0 & \frac{1}{4}w\gamma & 0 & 0 & 0 & \frac{wy}{4u} \\
\frac{1}{4}y & 0 & 0 & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & \frac{1}{4}y & 0 & 0 & 0 \\
\frac{1}{4}x & \frac{1}{4} & 0 & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & \frac{1}{4}x & \frac{1}{4} & 0 & 0 \\
\frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} \\
\frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x \\
\frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} \\
0 & 0 & \frac{1}{4} & \frac{1}{4}x & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4}x \\
0 & 0 & 0 & \frac{1}{4}y & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & 0 & 0 & \frac{1}{4}y \\
\frac{wy}{4u} & 0 & 0 & 0 & \frac{1}{4}w\gamma & 0 & \frac{1}{4}v\gamma & 0 & \frac{vy}{4u} & 0 & 0 & 0 \\
\frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} \\
0 & 0 & 0 & \frac{wy}{4u} & 0 & \frac{1}{4}w\gamma & 0 & \frac{1}{4}v\gamma & 0 & 0 & 0 & \frac{vy}{4u} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}x & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{vy}{4u} & \frac{1}{4}v\gamma & \frac{1}{4}w\gamma & \frac{wy}{4u} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}y & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{wy}{4u} & \frac{1}{4}w\gamma & \frac{1}{4}v\gamma & \frac{vy}{4u} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} a_{0,0} \\ a_{1,0} \\ a_{2,0} \\ a_{3,0} \\ a_{0,1} \\ a_{3,1} \\ a_{0,2} \\ a_{3,2} \\ a_{0,3} \\ a_{1,3} \\ a_{2,3} \\ a_{3,3} \end{bmatrix},
$$

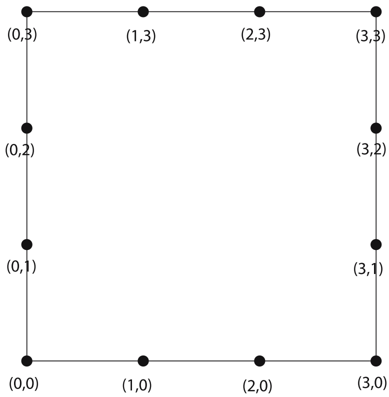

Figure 4.2: The control points projected on the original rectangle.

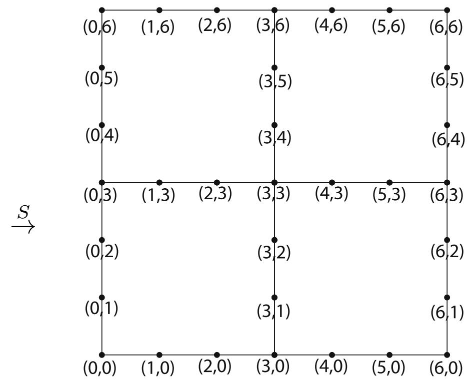

Figure 4.3: The projected control points after one subdivision.

where

$$\gamma := -\beta,\ v := u + 1,\ w := u - 1,\ x := 1 + u\beta,\ y := 2 + u\beta. \tag{4.2}$$

We denote the transformation matrix by S.

5 Local shape constrains.

We consider only the function case, where the starting data are values of f, p and q on the vertices of a rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$. We consider the one parameter family given by $\alpha = \frac{\beta}{4(1-\beta)}$ with $\beta \in [-1, 0)$. Corollary 3.8 implies C^1-convergence for any $\beta \in [-1, 0)$. We let $\lambda = u(1-\beta)$, where u is a free parameter. We also recall that for $\beta = -1$, the interpolant is the Sibson–Thomson element which is piecewise quadratic.

5.1 Positive interpolants.

We prove that if the control grid is positive, then the interpolant is positive. We use this result to give an algorithm to get a positive interpolant whenever the initial data make it possible.

PROPOSITION 5.1. *Suppose that $1 \le u \le -1/\beta$. If the initial control grid is positive, i.e.. $a_{k\ell} \ge 0$ for all k, ℓ, then the interpolant f is positive.*

PROOF. With the hypothesis $1 \le u \le -1/\beta$ and $-1 \le \beta < 0$ the quantities γ, v, w, x, y in (4.2) are nonnegative so that all entries in the matrix S in Proposition 4.1 are nonnegative. In the subdivision process we apply the matrix S recursively and it follows that all control coefficients on all levels are nonnegative. But then the values of the function f on $\cup(\mathcal{P}_n \times \mathcal{Q}_n) =$

$\mathcal{P} \times \mathcal{Q}$ are nonnegative. We have the result on $[a,b] \times [c,d]$ by continuous extension. □

We describe an algorithm to build a nonnegative interpolant on $[a,b] \times [c,d]$. Suppose that the initial data satisfy

$$
\begin{aligned}
&f(a,c) \geq 0 \quad \text{and} \quad (p(a,c) \geq 0, q(a,c) \geq 0 \quad \text{if } f(a,c) = 0),\\
&f(b,c) \geq 0 \quad \text{and} \quad (p(b,c) \leq 0, q(b,c) \geq 0 \quad \text{if } f(b,c) = 0),\\
&f(a,d) \geq 0 \quad \text{and} \quad (p(a,d) \geq 0, q(a,d) \leq 0 \quad \text{if } f(a,d) = 0),\\
&f(b,d) \geq 0 \quad \text{and} \quad (p(b,d) \leq 0, q(b,d) \leq 0 \quad \text{if } f(b,d) = 0).
\end{aligned}
\tag{5.1}
$$

ALGORITHM 5.1. *Let $h := b - a$, $k := d - c$ and choose $\lambda \geq 2$ such that*

$$
\begin{aligned}
a_{10} &= f(a,c) + h\tfrac{p(a,c)}{\lambda} \geq 0, & a_{01} &= f(a,c) + k\tfrac{q(a,c)}{\lambda} \geq 0,\\
a_{20} &= f(b,c) - h\tfrac{p(b,c)}{\lambda} \geq 0, & a_{31} &= f(b,c) + k\tfrac{q(b,c)}{\lambda} \geq 0,\\
a_{13} &= f(a,d) + h\tfrac{p(a,d)}{\lambda} \geq 0, & a_{02} &= f(a,d) - k\tfrac{q(a,d)}{\lambda} \geq 0,\\
a_{23} &= f(b,d) - h\tfrac{p(b,d)}{\lambda} \geq 0, & a_{32} &= f(b,d) - k\tfrac{q(b,d)}{\lambda} \geq 0.
\end{aligned}
\tag{5.2}
$$

Define $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$.
Perform HRC^1 defined by (2.1), (2.2), (2.3), *and* (2.4).

Since $\lambda \geq 2$, we obtain $\beta \in [-1, 0)$ so that the scheme is C^1-convergent. In view of (5.1), since $a_{00} = f(a,c) \geq 0$, $a_{30} = f(b,c) \geq 0$, $a_{33} = f(b,d) \geq 0$ and $a_{03} = f(a,d) \geq 0$ it is possible to choose $\lambda \geq 2$ so that the remaining control coefficients are nonnegative. By Proposition 5.1 the interpolant f is nonnegative.

EXAMPLE 5.1. *In Figure 5.1, we have computed three nonnegative HRC^1-interpolants choosing the same data on the vertices of $[0,1]^2$ except $p(0,1)$ which values are successively -1, -1.5 and -3. The values of λ are the smallest one so that* (5.2) *holds. In the first case (f_1, p_1, q_1), we have $\lambda = 2$ so that $\alpha = -1/8$ and $\beta = -1$ and we obtain the quadratic spline interpolant with piecewise linear derivatives. We see that the three interpolants are nonnegative.*

The plots of p and q in Figure 5.1 indicate that the regularity decreases with increasing λ, (see also Proposition 3.3). Thus one would normally choose the smallest $\lambda \geq 2$ in Algorithm 5.1.

5.2 Monotone interpolants.

We prove that if the control polygon is nondecreasing in the variable x then the interpolant is an nondecreasing function in x. We use this result to give an algorithm to get an nondecreasing interpolant in x as soon as the data make it possible.

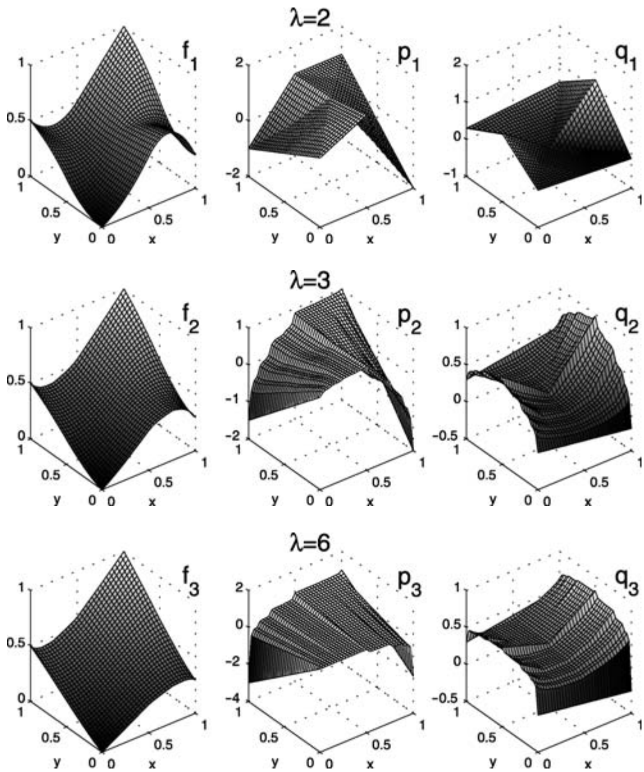

Figure 5.1: Nonnegative interpolants.

PROPOSITION 5.2. *Suppose that $u = -1/\beta$ with $\beta \in [-1, 0)$. If the initial grid is nondecreasing in x, i.e..*

$$\begin{cases} a_{00} \le a_{10} \le a_{20} \le a_{30}, \\ \qquad a_{01} \le a_{31}, \\ \qquad a_{02} \le a_{32}, \\ a_{03} \le a_{13} \le a_{23} \le a_{33}, \end{cases} \quad \textit{then} \quad \begin{cases} \bar a_{00} \le \bar a_{10} \le \bar a_{20} \le \bar a_{30} \le \bar a_{40} \le \bar a_{50} \le \bar a_{60}, \\ \qquad \bar a_{01} \le \bar a_{31} \le \bar a_{61}, \\ \qquad \bar a_{02} \le \bar a_{32} \le \bar a_{62}, \\ \bar a_{03} \le \bar a_{13} \le \bar a_{23} \le \bar a_{33} \le \bar a_{43} \le \bar a_{53} \le \bar a_{63}, \\ \qquad \bar a_{04} \le \bar a_{34} \le \bar a_{64}, \\ \qquad \bar a_{05} \le \bar a_{35} \le \bar a_{65}, \\ \bar a_{06} \le \bar a_{16} \le \bar a_{26} \le \bar a_{36} \le \bar a_{46} \le \bar a_{56} \le \bar a_{66} \end{cases}$$

at the first step, and the limit interpolant f is nondecreasing in x.

PROOF. We define (cf. Figures 4.2 and 4.3) horizontal differences $d_{i,j} := a_{i+1,j} - a_{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2$, $d_{0,j} := a_{3,j} - a_{0,j}$ for $j = 1, 2$, $\bar d_{i,j} := \bar a_{i+1,j} - \bar a_{i,j}$ for $i = 0, 1, \dots, 5$ and $j = 0, 3, 6$, and $\bar d_{i,j} := \bar a_{i+3,j} - \bar a_{i,j}$ for $i = 0, 3$ and $j = 1, 2, 4, 5$. We use the results of Proposition 4.1 and a computer algebra system to obtain:

$$
\begin{bmatrix}\bar{d}_{0,0}\\ \bar{d}_{1,0}\\ \bar{d}_{2,0}\\ \bar{d}_{3,0}\\ \bar{d}_{4,0}\\ \bar{d}_{5,0}\\ \bar{d}_{0,1}\\ \bar{d}_{3,1}\\ \bar{d}_{0,2}\\ \bar{d}_{3,2}\\ \bar{d}_{0,3}\\ \bar{d}_{1,3}\\ \bar{d}_{2,3}\\ \bar{d}_{3,3}\\ \bar{d}_{4,3}\\ \bar{d}_{5,3}\\ \bar{d}_{0,4}\\ \bar{d}_{3,4}\\ \bar{d}_{0,5}\\ \bar{d}_{3,5}\\ \bar{d}_{0,6}\\ \bar{d}_{1,6}\\ \bar{d}_{2,6}\\ \bar{d}_{3,6}\\ \bar{d}_{4,6}\\ \bar{d}_{5,6}\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4} & 0 & 0 & 0 & 0 & 0\\
-\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4} & 0 & 0 & 0 & 0 & 0\\
-\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4} & 0 & 0 & 0 & 0 & 0\\
\frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0\\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0\\
\frac{1-\beta}{4} & \frac{1-\beta}{8} & 0 & \frac{1-\beta}{8} & \frac{1+\beta}{8} & \frac{1+\beta}{4} & \frac{1+\beta}{8} & 0\\
0 & \frac{1-\beta}{8} & \frac{1-\beta}{4} & \frac{1-\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{4}\\
\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0\\
0 & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & 0\\
0 & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & 0\\
0 & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & 0\\
0 & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & 0\\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4}\\
\frac{1+\beta}{4} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1-\beta}{8} & \frac{1-\beta}{4} & \frac{1-\beta}{8} & 0\\
0 & \frac{1+\beta}{8} & \frac{1+\beta}{4} & \frac{1+\beta}{8} & \frac{1-\beta}{8} & 0 & \frac{1-\beta}{8} & \frac{1-\beta}{4}\\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2}\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4}\\
0 & 0 & 0 & 0 & 0 & -\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4}\\
0 & 0 & 0 & 0 & 0 & -\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4}\\
0 & 0 & 0 & 0 & 0 & \frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}d_{0,0}\\ d_{1,0}\\ d_{2,0}\\ d_{0,1}\\ d_{0,2}\\ d_{0,3}\\ d_{1,3}\\ d_{2,3}\end{bmatrix}.
$$

The hypothesis implies that $d_{k\ell} \geq 0$. Since $-1 \leq \beta < 0$, we obtain $\bar{d}_{ij} \geq 0$.

We can extend the result recursively on each sub rectangle of $\mathcal{P}_n \times \mathcal{Q}_n$. At each step the control grid is nondecreasing in the direction x so that the function p is nonnegative on $\cup(\mathcal{P}_n \times \mathcal{Q}_n) = \mathcal{P} \times \mathcal{Q}$. By continuous extension p is nonnegative on $[a,b] \times [c,d]$ and f is nondecreasing in x. □

We define an algorithm to construct an interpolant on $[a,b] \times [c,d]$ that is nondecreasing in x. Suppose that the initial data satisfy

$$
\begin{aligned}
&p(a,c) \geq 0,\ p(b,c) \geq 0,\ f(a,c) < f(b,c) \quad \text{and}\\
&p(a,d) \geq 0,\ p(b,d) \geq 0,\ f(a,d) < f(b,d).
\end{aligned}
$$

Algorithm 5.2. *Let $h := b - a, k =: d - c$ and choose $\lambda \geq 2$ such that*

$$(5.3)\quad \begin{aligned} a_{10} = f(a,c) + h\tfrac{p(a,c)}{\lambda} &\leq a_{20} = f(b,c) - h\tfrac{p(b,c)}{\lambda},\\ a_{01} = f(a,c) + k\tfrac{q(a,c)}{\lambda} &\leq a_{31} = f(b,c) + k\tfrac{q(b,c)}{\lambda},\\ a_{02} = f(a,d) - k\tfrac{q(a,d)}{\lambda} &\leq a_{32} = f(b,d) - k\tfrac{q(b,d)}{\lambda},\\ a_{13} = f(a,d) + h\tfrac{p(a,d)}{\lambda} &\leq a_{23} = f(b,d) - h\tfrac{p(b,d)}{\lambda}. \end{aligned}$$

Define $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$.
Perform HRC^1 defined by (2.1), (2.2), (2.3), *and* (2.4).

Since the β used in this algorithm always belongs to the interval $[-1, 0)$ the interpolating scheme is C^1-convergent. Morover, since $a_{00} \leq a_{10}$, $a_{20} \leq a_{30}$, $a_{03} \leq a_{13}$, $a_{23} \leq a_{33}$, the control grid and the interpolant f are nondecreasing in the variable x.

Example 5.2. *In the 3 following pictures (Figure 5.2), we have computed three interpolants which are monotone in the x-direction. We choose the same data on the vertices of* $[0,1]^2$ *except* $p(1,0)$ *which values are successively* 0.3, 0.9 *and* 1.8. *The values of λ are the smallest one satisfying* (5.3). *In the first case*

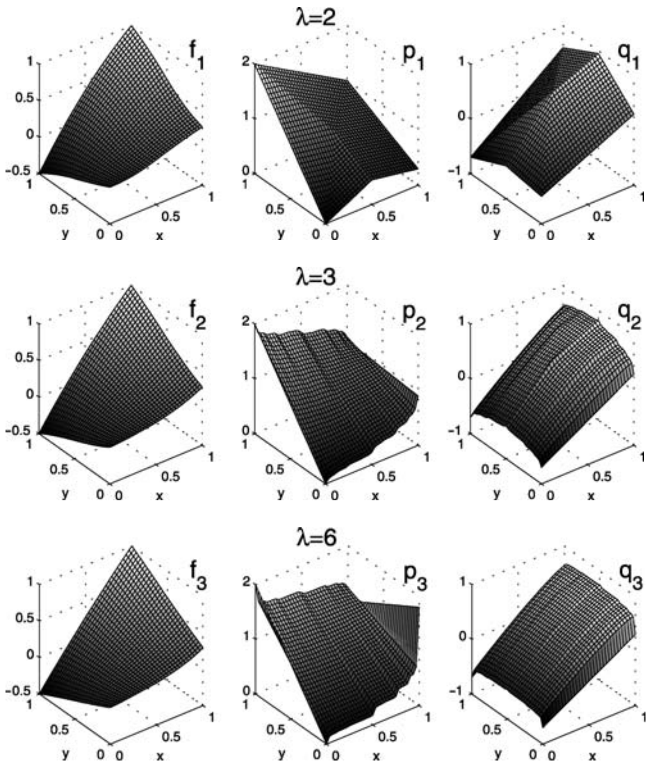

Figure 5.2: Increasing interpolants in the variable x.

(f_1, p_1, q_1), we have $\lambda = 2$ so that $\alpha = -1/8$ and $\beta = -1$ and we obtain the quadratic spline interpolant with piecewise linear derivatives. We observe that the three derivatives p_1, p_2, p_3 are positive.

Obviously, there is a similar algorithm for achieving monotonicity in the y-direction. We can also combine different constraints. For example if the data are positive and nondecreasing in the x-direction, we can obtain a positive and nondecreasing interpolants by choosing the largest λ in Algorithms 5.1 and 5.2.

6 Examples of global constrains.

6.1 The first example.

We start with the grid defined by

$$\{(x_i, y_j)\} = \{0, 0.25, 0.7, 0.92, 1\} \times \{0, 0.2, 0.6, 1\}$$

and sub-rectangles $R_{i,j}, i = 1, \ldots, 4, j = 1, \ldots, 3$. The initial data for the function f at the vertices of the grid are

$x\backslash y$	0	0.2	0.6	1
0	0	−0.9511	0.5878	0.0000
0.25	0.0625	−0.8886	0.6503	0.0625
0.7	0.4900	−0.4611	1.0778	0.4900
0.92	0.8464	−0.1047	1.7000	1.7000
1	1.0000	0.0489	1.7000	1.7000

They are stricly increasing along x except that $f(0.92, 0.6) = f(1, 0.6) = f(0.92, 1) = f(1, 1)$. Since the initial data where sampled from the function $f(x, y) = x^2 - \sin(2\pi y)$, we compute the exact derivatives p and q and we add a random number in $[0, 0.2]$ for p. We have an exception for $R_{4,3}$. We choose the example:

p

$x\backslash y$	0	0.2	0.6	1
0	0.0388	0.1098	0.1255	0.1675
0.25	0.6810	0.6863	0.6398	0.5743
0.7	1.5138	1.4670	1.4794	1.4851
0.92	1.9664	1.9711	0	0
1	2.0469	2.0784	0	0

q

$x\backslash y$	0	0.2	0.6	1
0	−6.2832	−5.0832	1.9416	6.2832
0.25	−6.2832	−5.0832	1.9416	6.2832
0.70	−6.2832	−5.0832	1.9416	6.2832
0.92	−6.2832	−5.0832	0	0
1	−6.2832	−5.0832	0	0

All the derivatives $p = f_x$ are nonnegative except $p(0.92, 0.6) = p(1, 0.6) = p(0.92, 1) = p(1, 1) = 0$. We add $q(0.92, 0.6) = q(1, 0.6) = q(0.92, 1) = q(1, 1) = 0$.

On each subrectangle $R_{i,j}$, we compute the smallest $\lambda_{i,j} \geq 2$ which gives an nondecreasing control grid in the variable x. For the rectangle $R_{4,3}$, we can built a constant interpolant with any $\lambda_{4,3}$. Let us choose $\lambda_{4,3} = 2$. Then we compute $\lambda = \max \lambda_{i,j} = 3.1844$, $\beta = \frac{1}{1-\lambda} = -0.4578$ and $\alpha = \frac{\beta}{4(1-\beta)} = -0.0785$. On each sub-rectangle, we perform HRC^1 defined by (2.1), (2.2), (2.3), and (2.4). See Figure 6.1.

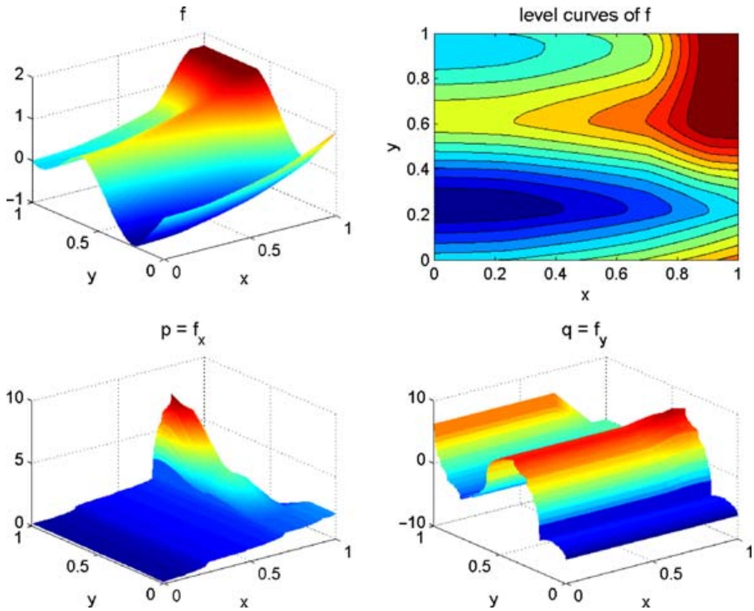

Figure 6.1: Nondecreasing interpolant in the variable x on a mesh with, $\lambda = 3.1843$.

6.2 The second example.

This example was proposed in [2]. The initial grid is $\{-0.07, 0.33, 0.55, 0.69, 0.84, 0.93, 0.98, 1.02, 1.08, 1.13\} \times \{-2.3, -1.61, -0.92, -0.51, -0.22, 0.0\}$, and we will use the sub-rectangles $R_{i,j}, i = 1, \ldots, 9, j = 1, \ldots, 5$. The given data $f^0_{i,j}$ of the function f are

$x \backslash y$	-2.3	-1.61	-0.92	-0.51	-0.22	0
-0.07	-34.54	-13.82	-10.10	-7.26	-5.66	-4.53
0.33	-34.54	-13.82	-10.10	-7.26	-5.66	-4.13
0.55	-34.54	-13.82	-10.10	-7.26	-4.88	-3.35
0.69	-34.54	-13.82	-10.10	-4.82	-3.34	-2.73
0.84	-34.54	-13.82	-2.52	-2.22	-1.98	-1.78
0.93	-34.54	-2.68	-1.88	-1.56	-1.41	-1.28
0.98	-3.06	-2.28	-1.63	-1.32	-1.15	-1.05
1.02	-2.86	-1.92	-1.39	-1.10	-0.92	-0.81
1.08	-2.37	-1.60	-1.17	-0.90	-0.72	-0.60
1.13	-1.89	-1.30	-0.95	-0.71	-0.54	-0.41

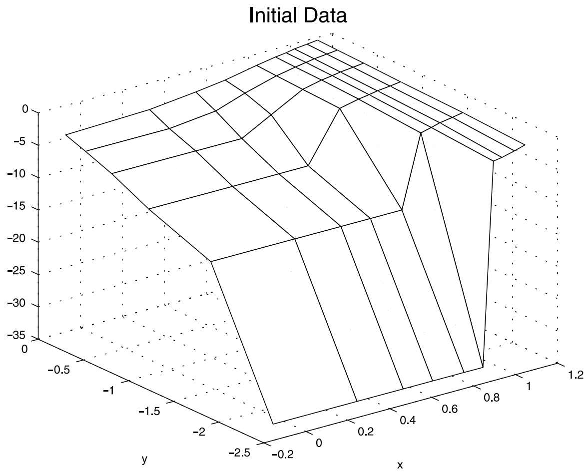

Figure 6.2: Initial mesh and function.

The data are nondecreasing in the directions x and y (see Figure 6.2) so that we will choose non negative derivatives p and q to get a nondecreasing interpolant in both directions. Notice that if $f^0_{i,j} = f^0_{i+1,j}$, we must choose $p^0_{i,j} = p^0_{i+1,j} = 0$ and $q^0_{i,j} = q^0_{i+1,j}$ and similarly on the other direction. With this exception, we can choose any non negative derivatives $p^0_{i,j}$ and $q^0_{i,j}$ to get an nondecreasing interpolant in both directions. Again on each sub-rectangle $R_{i,j}$, we compute the smallest $\lambda_{i,j} \geq 2$ which gives an nondecreasing control grid in the variable x and in the variable y. Then we compute $\lambda = \max \lambda_{i,j}$, $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$. On each sub-rectangle, we perform HRC^1 defined by (2.1), (2.2), (2.3), and (2.4).

Case 1: We have computed the initial derivatives $p^0_{i,j}$ and $q^0_{i,j}$ using the standard two point forward differences. The computed value is $\lambda = 4.5455$. See Figure 6.3.

Case 2: We took random positive derivatives (between 0 and 2). The computed value is $\lambda = 4.9861$. See Figure 6.4.

It is remarkable that we can obtain a monotone interpolant even with randomly chosen derivatives. Also the graphs of the functions look very similar in Figures 6.3 and 6.4. The differences are mainly in the y-derivative q.

7 Final remarks.

1. In the shape preserving algorithms the subdivision was carried out using the HRC^1-algorithm. The control coefficients were used only to choose parameters to ensure a final interpolant with the desired shape.
2. By applying Proposition 4.1 it is possible to reformulate the HRC^1 scheme as a stationary subdivision scheme working on points in $\mathbb{R}^s$. We start with 12

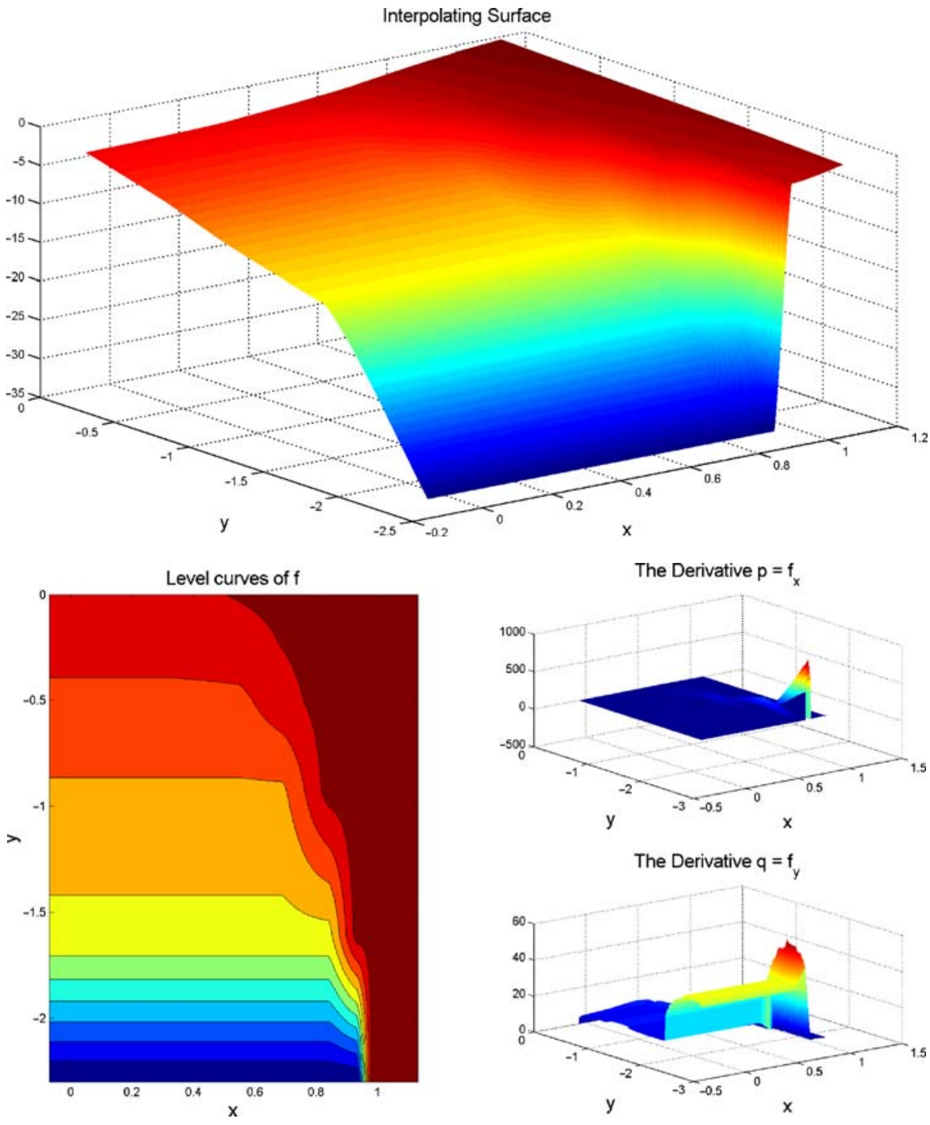

Figure 6.3: Nondecreasing interpolant using forward differences to estimate derivatives.

control coefficients $a_{0,0}, a_{1,0}, a_{2,0}, a_{3,0}$, $a_{0,1}, a_{3,1}, a_{0,2}, a_{3,2}, a_{0,3}, a_{1,3}, a_{2,3}, a_{3,3}$ in $\mathbb{R}^s$, $s \geq 1$, (α, β) in the convergence region in Figure 3.1, and $\lambda \geq 2$. Under suitable restrictions on the "rectangular structure" of the initial control coefficients we could then define an algorithm SRC^1 based on recursively using the matrix S. However we will not consider this any further here.

3. We note that S has negative minors and thus is not a totally positive matrix. For example the 2×2 minor constructed from the entries in rows 2 and 8 and columns 1 and 2 has the value $-1/4$ for all values of α, β, and λ.

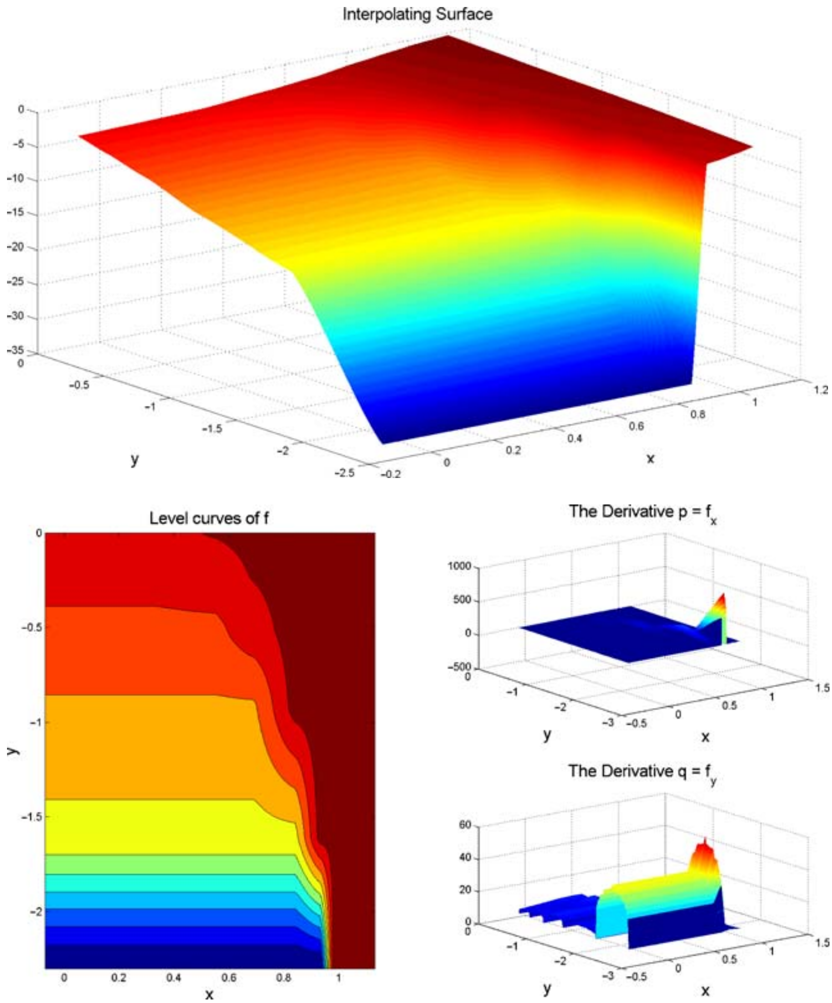

Figure 6.4: Nondecreasing interpolant with random derivatives.

4. Unfortunately, the algorithm HRC^1 is in general not able to give a convex interpolant when starting with convex data. To see this we consider the function given by

$$\phi(x,y) := \begin{cases} (1-x-y)^3 & \text{if } 1-x-y \geq 0 \\ 0 & \text{if } 1-x-y \leq 0. \end{cases}$$

This function is C^2 and convex on $[0,1]^2$. To construct a convex HRC^1-interpolant f we note that f must be convex along the diagonal $\delta := \{(x,y) \in [0,1]^2 : x+y=1\}$. Since ϕ and its partial derivatives vanish at the two corners $(1,0)$ and $(0,1)$ the same holds true for f. This means that f must vanish identically on δ. We now show that this is not possible regardless of how we choose α and β.

At step 0, we sample the function and its derivatives on the vertices of the square $[0,1]^2$ and we obtain $f^0_{i,j} = p^0_{i,j} = q^0_{i,j} = 0$ for $(i,j) \neq (0,0)$ and $f^0_{0,0} = 1$, $p^0_{0,0} = q^0_{0,0} = -3$. Using (2.4) we compute the values at the midpoint $(1/2, 1/2)$. We find $f^1_{11} = \frac{1}{4} + 3\alpha$ and $p^1_{11} = q^1_{11} = -\frac{1}{2} - \beta$. Convexity on δ implies that $\alpha = -1/12$ and $\beta = -1/2$. Moreover for these values of the parameters we must have $f^n_{ij} = 0$ for all points on δ. But already the value f^2_{31} at the point $(3/4, 1/4)$ on δ is nonzero. To see this we first compute $f^1_{1,0} = 1/4$, $p^1_{1,0} = -3/4$, $q^1_{1,0} = -3/2$ by (2.2), and then $f^1_{2,1} = p^1_{2,1} = q^1_{2,1} = 0$ by (2.3). We now find $f^2_{3,1} = 1/64 \neq 0$ using (2.4). Thus the HRC^1-interpolant is not convex.

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