A Generalized Taylor Factorization for Hermite Subdivisions Schemes

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Abstract

In a recent paper, we investigated factorization properties of Hermite subdivision schemes by means of the so–called *Taylor factorization*. This decomposition is based on a spectral condition which is satisfied for example by all interpolatory Hermite schemes. Nevertheless, there exist examples of Hermite schemes, especially some based on cardinal splines, which fail the spectral condition. For these schemes (and others) we provide the concept of an *generalized Taylor factorization* and show how it can be used to obtain convergence criteria for the Hermite scheme by means of factorization and contractivity.

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1 Introduction

Hermite subdivision schemes operate by applying matrix schemes to vector data where the entries of the vector represent the value of an underlying function and a consecutive chain of derivatives of that function. As appropriate notational convention for these objects, vectors in $\mathbb{R}^r$ will be labeled by lowercase boldface letters as $\boldsymbol{y} = [y_j]_{j=1,\dots,r}$ and matrices in $\mathbb{R}^{r\times r}$ will be written as uppercase boldface letters, like $\boldsymbol{A} = [a_{jk}]_{j,k=1,\dots,r}$.

By $\boldsymbol{A} \in \ell^{r\times r}(\mathbb{Z})$, we also denote a multiindexed sequence of matrices, that is, for all $\alpha \in \mathbb{Z}$ the sequence element $\boldsymbol{A}(\alpha) = [a_{jk}(\alpha)]_{j,k=1,\dots,r}$ is an $r \times r$ matrix. Any such sequence will be called a *mask* provided that it is finitely supported, that is, there exists $N \in \mathbb{N}$ such that

$$\operatorname{supp} \boldsymbol{A} := \{\alpha \in \mathbb{Z} \,:\, \boldsymbol{A}(\alpha) \neq 0\} \subseteq [-N, N].$$

To any mask $\boldsymbol{A}$ we associate the *stationary subdivision operator* $S_{\boldsymbol{A}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$, defined as

$$S_{\boldsymbol{A}}\boldsymbol{c}(\alpha) := \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\, \boldsymbol{c}(\beta), \qquad \boldsymbol{c} \in \ell^r(\mathbb{Z}). \tag{1}$$

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Now we define the *Hermite subdivision scheme* $H_{\boldsymbol{A}}$. Starting with $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$, for $n \in \mathbb{N}_0$, we define $\boldsymbol{f}_{n+1} \in \ell^{d+1}(\mathbb{Z})$ by

$$\boldsymbol{D}^{n+1} \boldsymbol{f}_{n+1}(\alpha) = S_{\boldsymbol{A}} \boldsymbol{D}^n \boldsymbol{f}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\, \boldsymbol{D}^n\, \boldsymbol{f}_n(\beta), \tag{2}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^d} \end{bmatrix}$$

is the diagonal matrix with diagonal entries 2^{-j}, respectively, $j = 0, \dots, d$.

For the iterated application of the Hermite subdivision scheme, the first component $\boldsymbol{f}_n^{(0)}(\beta)$ can be interpreted as the value of a function ϕ_n at $\beta/2^n$, while the next one, $\boldsymbol{f}_n^{(1)}(\beta)$, describes the derivative $\phi_n'(\beta/2^n)$, and so on, up to the last one, $\boldsymbol{f}_n^{(d)}(\beta)$ which is $\phi_n^{(d)}(\beta/2^n)$,

2 The Taylor factorization

To introduce the definition of the spectral condition, we associate to any function $f \in C^d(\mathbb{R})$ the vector sequence $\boldsymbol{v}_f \in \ell^{d+1}(\mathbb{Z})$ with

$$\boldsymbol{v}_f(\alpha) := \begin{bmatrix} f(\alpha) \\ f'(\alpha) \\ \vdots \\ f^{(d)}(\alpha) \end{bmatrix}, \qquad \alpha \in \mathbb{Z}. \tag{3}$$

A fundamental property of Hermite subdivision schemes is the *spectral condition* introduced in [3], which requires the existence of particular *polynomial* eigenvalues of the stationary subdivision operator $S_{\boldsymbol{A}}$.

Definition 1 *A mask $\boldsymbol{A}$ or its associated subdivision operator $S_{\boldsymbol{A}}$ satisfies the* spectral condition *of order d if there exist polynomials $p_j \in \Pi_j$, $\deg p_j = j$, $j = 0, \dots, d$, such that*

$$S_{\boldsymbol{A}} \boldsymbol{v}_j = \frac{1}{2^j} \boldsymbol{v}_j, \qquad \boldsymbol{v}_j := \boldsymbol{v}_{p_j} = \begin{bmatrix} p_j \\ p_j' \\ \vdots \\ p_j^{(d)} \end{bmatrix}, \tag{4}$$

where we will always assume that p_j is normalized such that $p_j(x) = \frac{1}{j!} x^j + \cdots$.

It was proved in [3] that the spectral condition is also equivalent to the *sum rule* introduced by Bin Han et al [6, 7].

Definition 2 *The* Taylor operator T_d *and the* complete Taylor operator $\widetilde{T}_d$ *of order* d, *acting on* $\ell^{(d+1)}(\mathbb{Z})$ *are*

$$T_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix}, \quad \widetilde{T}_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix}.$$

The following factorization result has been proved, among others, in [9]. We also mention that, provided a factorization exists, the convergence of the Hermite subdivision scheme $H_{\boldsymbol{A}}$ is equivalent to the *contractivity* of the stationary vector subdivision scheme $S_{\widetilde{\boldsymbol{B}}}$.

Theorem 3 *If the mask* $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *satisfies the spectral condition of order* d, *then there exist two finitely supported masks* $\boldsymbol{B}$ *and* $\widetilde{\boldsymbol{B}}$ *in* $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *such that*

$$T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d, \qquad \textit{and} \qquad \widetilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}} \widetilde{T}_d. \tag{5}$$

We conclude this section by giving two examples of Hermite subdivision schemes and their associated complete factorizations.

2.1 An Interpolatory Scheme for $d = 1$

We recall the interpolatory Hermite subdivision scheme HC^1, proposed by Merrien [8]. The convergence is detailled in [1]. The mask has support included in $[-1, 1]$ with:

$$\boldsymbol{A}(-1) = \frac{1}{4}\begin{bmatrix} 2 & 4\lambda \\ 2(1-\mu) & \mu \end{bmatrix}, \ \boldsymbol{A}(0) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{A}(1) = \frac{1}{4}\begin{bmatrix} 2 & -4\lambda \\ -2(1-\mu) & \mu \end{bmatrix}.$$

For every choice of λ and μ, the scheme $H_{\boldsymbol{A}}$ reproduces affine functions so that it satisfies the spectral condition with $p_0(x) = 1$, $p_1(x) = x$. According to Theorem 3, there exists a *complete Taylor (vector subdivision) scheme* $S_{\widetilde{\boldsymbol{B}}}$ associated with $H_{\boldsymbol{A}}$.

Since $\widetilde{T}_1 = \begin{bmatrix} \Delta & -1 \\ 0 & \Delta \end{bmatrix}$, the nonzero matrices of the mask of the complete Taylor scheme are found to be

$$\widetilde{\boldsymbol{B}}(0) = \begin{bmatrix} 1 & 2\lambda \\ (1-\mu) & \mu/2 \end{bmatrix}, \qquad \widetilde{\boldsymbol{B}}(1) = \begin{bmatrix} \mu & -\mu/2 - 2\lambda \\ \mu - 1 & 1 - \mu/2 \end{bmatrix}.$$

2.2 A Non Interpolatory Scheme for $s = 1$ and $d = 1$

We build the de Rham transform [2] of the previous scheme $H_{\boldsymbol{A}}$ with support in $[-2, 1]$. The respective non zero matrices are

$$\begin{aligned}
\boldsymbol{A}(-2) &= \frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & 4\lambda+2\lambda\mu \\ 4-2\mu-2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}, \\
\boldsymbol{A}(-1) &= \frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & 8\lambda-2\lambda\mu \\ 4-2\mu-2\mu^2 & 2\mu+\mu^2-8\lambda(1-\mu) \end{bmatrix}, \\
\boldsymbol{A}(0) &= \frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & -8\lambda+2\lambda\mu \\ -4+2\mu+2\mu^2 & 2\mu+\mu^2-8\lambda(1-\mu) \end{bmatrix}, \\
\boldsymbol{A}(1) &= \frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & -4\lambda-2\lambda\mu \\ -4+2\mu+2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix},
\end{aligned}$$

and $H_{\boldsymbol{A}}$ satisfies the spectral condition with $p_0(x) = 1$, $p_1(x) = x - 1/2$. A complete Taylor scheme $S_{\widetilde{\boldsymbol{B}}}$ is associated with $H_{\boldsymbol{A}}$ and its mask has the following nonzero coefficients:

$$\begin{aligned}
\widetilde{\boldsymbol{B}}(-1) &= \frac{1}{4}\begin{bmatrix} 2+4\lambda(1-\mu) & 2\lambda(2+\mu) \\ 4-2\mu-2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}, \\
\widetilde{\boldsymbol{B}}(0) &= \frac{1}{4}\begin{bmatrix} 2\mu+2\mu^2-8\lambda(1-\mu) & -4\lambda(1-\mu)-\mu^2 \\ 0 & 2\mu-16\lambda(1-\mu) \end{bmatrix}, \\
\widetilde{\boldsymbol{B}}(1) &= \frac{1}{4}\begin{bmatrix} -2+4\lambda(1-\mu)+2\mu+2\mu^2 & -6\lambda\mu-\mu^2-2\mu+2 \\ -4+2\mu+2\mu^2 & 4-2\mu-\mu^2+8\lambda(1-\mu) \end{bmatrix}.
\end{aligned}$$

3 Cardinal spline functions

Cardinal splines, i.e., splines with integer knots, are a classical concept in Approximation theory, cf. [13], which anticipated a lot of the later theory of approximation by translation invariant spaces. Let us briefly recall some basics and settle the notation.

3.1 Construction

Our presentation is based on a construction detailed by Michelli in [10]. Let

$$\varphi_0(x) = \chi_{[0,1]} = \begin{cases} 1 \text{ if } x \in [0,1], \\ 0 \text{ if } x \notin [0,1]. \end{cases}$$

For $r = 1, 2, \ldots$, we build φ_r by means of autoconvolution as $\varphi_r = \varphi_0 * \varphi_{r-1}$ or $\varphi_r(x) = \int_{x-1}^{x} \varphi_{r-1}(t)dt$.

By recursion, it is easily seen that $\sigma_r := \operatorname{supp}(\varphi_r) = [0, r+1]$, that φ_r is a $\mathcal{C}^{r-1}$ piecewise polynomial of degree r and that the functions $\varphi_r(\cdot - \alpha)$, $\alpha \in \mathbb{Z}$, form a nonnegative partition of unity, i.e. $\sum_{\alpha\in\mathbb{Z}} \varphi_r(x-\alpha) = 1$ and $\varphi_r(x-\alpha) \geq 0$. Moreover,

$$\varphi_r(x) = \frac{1}{2^r}\sum_{\alpha\in\mathbb{Z}} \binom{r+1}{\alpha} \varphi_r(2x-\alpha), \qquad \binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & \text{if } 0 \leq j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Considering $v(x) = \sum_{\alpha\in\mathbb{Z}} f_0^{(0)}(\alpha)\varphi_r(x-\alpha)$, which is a finite sum for any $x \in \mathbb{R}$ since $\varphi_r(x-\alpha) = 0$ if $x-\alpha \notin [0, r+1]$, we deduce for $n \in \mathbb{N}_0$ that $v(x) = \sum_{\alpha\in\mathbb{Z}} f_n^{(0)}(\alpha)\varphi_r(2^n x - \alpha)$ where

$$f_{n+1}^{(0)}(\alpha) = \frac{1}{2^r}\sum_{\beta\in\mathbb{Z}}\binom{r+1}{\alpha-2\beta} f_n^{(0)}(\beta) =: \sum_{\beta\in\mathbb{Z}} a_r(\alpha-2\beta) f_n^{(0)}(\beta), \qquad \alpha \in \mathbb{Z}, \tag{6}$$

that is,

$$a_r(\alpha) = \frac{1}{2^r}\binom{r+1}{\alpha}, \qquad \alpha \in \mathbb{Z}. \tag{7}$$

Moreover, the well–known derivative formula for cardinal B–spline yields

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} 2^{ni}\Delta^i f_n^{(0)}(\alpha-i)\,\varphi_{r-i}\left(2^n x-\alpha\right), \qquad i = 0,\dots,r-1. \tag{8}$$

We have a particular case when $i = r-1$. Since the function φ_1 is piecewise linear with $\varphi_1(\alpha) = \delta_{1\alpha}$, we obtain $\dfrac{d^{r-1}v}{dx^{r-1}}(\beta/2^n) = 2^{ni}\Delta^i f_n^{(0)}(\beta-r+1)$.

We conclude this short presentation by a theorem of convergence (Theorem 2.2 page 63 in [10]): there exists a constant e_r such that

$$\begin{cases} \left|v(\alpha/2^n) - f_n^{(0)}(\alpha)\right| \le \frac{r-1}{2^n}\|\Delta v_0\|_\infty \\[2ex] \left|v\left(\frac{\alpha}{2^n}+\frac{r+1}{2^{n+1}}\right) - f_n^{(0)}(\alpha)\right| \le \frac{e_r}{4^n}\|\Delta^2 v_0\|_\infty \end{cases}$$

3.2 A Hermite subdivision scheme

Our goal is to give an example of a convergent Hermite subdivision scheme that does not satisfy the spectral condition, hence does not admit a Taylor factorization as described in the preceding section. To that end, let us choose $r > d$. With (6), we have defined a scalar subdivision scheme, and the mask $\{a_r(\alpha)\}$ from (7) has support $[0, r+1]$

We define a Hermite subdivision scheme of order d with mask $\{A(\alpha)\}$ and support $[0, r+d+1]$ by applying differences to the mask a_r, yielding

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} a_r(\alpha) & 0 & \dots & 0 \\ \Delta a_r(\alpha-1) & 0 & \dots & 0 \\ \vdots & & & \\ \Delta^d a_r(\alpha-d) & 0 & \dots & 0 \end{bmatrix}.$$

We begin with $f_0 \in \ell^{d+1}(\mathbb{Z})$ and $f_n \in \ell^{d+1}(\mathbb{Z})$ defined by (2) and notice that $2^{-(n+1)} f_{n+1}^{(1)}(\alpha) = \sum_{\beta\in\mathbb{Z}} \Delta a_r(\alpha-1-2\beta) f_n^{(0)}(\beta) = \Delta f_{n+1}^{(0)}(\alpha-1)$ so that for $n \ge 1$,

$$f_n^{(1)}(\alpha) = 2^n \Delta f_n^{(0)}(\alpha-1). \tag{9}$$

Similarly for $i = 2,\dots,d$:

$$f_n^{(i)}(\alpha) = 2^{in}\Delta^i f_n^{(0)}(\alpha-i). \tag{10}$$

Now with (6) and (8), for $n > 0$,

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} f_n^{(i)}(\alpha)\varphi_{r-i}(2^n x - \alpha), \qquad i = 0, \dots, d.$$

3.3 Spectral properties for $r = 3$

For $r = 3$ and $d = 2$, the first row of the mask $A(\alpha)$ is given by

α	0	1	2	3	4	5	6
$8a_{00}(\alpha)$	1	4	6	4	1	0	0
$8a_{10}(\alpha)$	1	3	2	-2	-3	-1	0
$8a_{20}(\alpha)$	1	2	-1	-4	-1	2	1

Let p_k, $k = 0, 1, 2$, be defined by $p_0(x) = 1$, $p_1(x) = x + 2$ and $p_3(x) = 1/2(x^2 + 4x + 11/3)$. One can check that $\sum_{\beta\in\mathbb{Z}} a_3(\alpha - 2\beta)p_k(\beta) = 2^{-k}p_k(\alpha)$, and with (9) and (10), that

$$f_0(\alpha) = \begin{bmatrix} p_k(\alpha) \\ p_k'(\alpha) \\ p_k''(\alpha) \end{bmatrix} \qquad \Rightarrow \qquad f_1(\alpha) = 1/2^i \begin{bmatrix} p_k(\alpha) \\ 2\Delta p_k(\alpha - 1) \\ 4\Delta^2 p_k(\alpha - 2) \end{bmatrix}.$$

Since $\Delta p_0(x) = p_0'(x) = 0 = \Delta^2 p_0(x) = p_0''(x)$, $\Delta p_1(x) = p_1'(x) = 1$, $\Delta^2 p_1(x) = p_1''(x) = 0$ and $\Delta p_2(x) = x + 5/2 \neq p_2'(x) = x + 2$, $\Delta^2 p_2(x) = p_2''(x) = 1$, and since we have $Df_1(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta)f_0(\beta)$, we obtain

$$\sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) \begin{bmatrix} p_0(\beta) \\ p_0'(\beta) \\ p_0''(\beta) \end{bmatrix} = \begin{bmatrix} p_0(\alpha) \\ p_0'(\alpha) \\ p_0''(\alpha) \end{bmatrix}$$

$$\sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) \begin{bmatrix} p_1(\beta) \\ p_1'(\beta) \\ p_1''(\beta) \end{bmatrix} = 1/2 \begin{bmatrix} p_1(\alpha) \\ p_1'(\alpha) \\ p_1''(\alpha) \end{bmatrix}$$

but

$$\sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) \begin{bmatrix} p_2(\beta) \\ p_2'(\beta) \\ p_2''(\beta) \end{bmatrix} = 1/4 \begin{bmatrix} p_2(\alpha) \\ \alpha + 3/2 \\ p_2''(\alpha) \end{bmatrix} \neq 1/4 \begin{bmatrix} p_2(\alpha) \\ p_2'(\alpha) \\ p_2''(\alpha) \end{bmatrix},$$

hence, the spectral condition is not satisfied for the scheme.

Nevertheless, if $\boldsymbol{R}$ is the triangular matrix in $\mathbb{R}^{3\times 3}$ defined by

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix},$$

we easily check that

$$\sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta)\boldsymbol{R} \begin{bmatrix} p_k(\beta) \\ p_k'(\beta) \\ p_k''(\beta) \end{bmatrix} = 1/2^k \boldsymbol{R} \begin{bmatrix} p_k(\alpha) \\ p_k'(\alpha) \\ p_2''(\alpha) \end{bmatrix}, \qquad k = 0, 1, 2.$$

4 Generalized Taylor factorization

Motivated by the above example, we slightly extend the concept of the spectral condition.

Definition 4 *A mask $\boldsymbol{A}$ or its associated subdivision operator $S_{\boldsymbol{A}}$ satisfies the* generalized spectral condition *of order d if there exist a nonsingular matrix $\boldsymbol{R} \in \mathbb{R}^{(d+1)\times(d+1)}$ and polynomials $p_j \in \Pi_j$, $\deg p_j = j$, $j = 0, \dots, d$, such that*

$$S_{\boldsymbol{A}} \boldsymbol{R} \boldsymbol{v}_j = \frac{1}{2^j} \boldsymbol{R} \boldsymbol{v}_j, \qquad \boldsymbol{v}_j := \boldsymbol{v}_{p_j} = \begin{bmatrix} p_j \\ p_j' \\ \vdots \\ p_j^{(d)} \end{bmatrix}, \tag{11}$$

where we will always assume that p_j is normalized such that $p_j(x) = \frac{1}{j!} x^j + \cdots$.

Theorem 5 *If the mask $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ satisfies the generalized spectral condition of order d, then there exist two finitely supported masks $\boldsymbol{B}_R$ and $\widetilde{\boldsymbol{B}}_R$ in $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that*

$$T_d \boldsymbol{R}^{-1} S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}_R} T_d \boldsymbol{R}^{-1} \qquad \textit{and} \qquad \widetilde{T}_d \boldsymbol{R}^{-1} S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}_R} \widetilde{T}_d \boldsymbol{R}^{-1}. \tag{12}$$

Proof: Let us define $\boldsymbol{A}_R = \boldsymbol{R}^{-1} \boldsymbol{A} \boldsymbol{R}$ or equivalently, for any $\alpha \in \mathbb{Z}$, $\boldsymbol{A}_R(\alpha) = \boldsymbol{R}^{-1} \boldsymbol{A}(\alpha) \boldsymbol{R}$. Then for any $i = 0, \dots, d$, we obtain

$$\begin{aligned} & \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta) \boldsymbol{R} \boldsymbol{v}_i(\beta) = \frac{1}{2^i} \boldsymbol{R} \boldsymbol{v}_i(\alpha) \\ \Leftrightarrow & \sum_{\beta\in\mathbb{Z}} \boldsymbol{R} \boldsymbol{A}_R(\alpha - 2\beta) \boldsymbol{v}_i(\beta) = \frac{1}{2^i} \boldsymbol{R} \boldsymbol{v}_i(\alpha) \\ \Leftrightarrow & \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}_R(\alpha - 2\beta) \boldsymbol{v}_i(\beta) = \frac{1}{2^i} \boldsymbol{v}_i(\alpha). \end{aligned}$$

Hence the scheme $S_{\boldsymbol{A}_R}$ satisfies the spectral condition. Thus, by Theorem 3, there exist two finitely supported masks $\boldsymbol{B}_R$ and $\widetilde{\boldsymbol{B}}_R$ in $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $T_d S_{\boldsymbol{A}_R} = 2^{-d} S_{\boldsymbol{B}_R} T_d$ and $\widetilde{T}_d S_{\boldsymbol{A}_R} = 2^{-d} S_{\widetilde{\boldsymbol{B}}_R} \widetilde{T}_d$ which can be written as (12). □

5 Convergence

We begin by recalling some basic convergence concepts related to Hermite subdivision schemes and their associated stationary schemes. These notions have been introduced and investigated in [9].

Definition 6 *Let $\boldsymbol{B} \in \ell^{r\times r}(\mathbb{Z})$ be a mask and $S_{\boldsymbol{B}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$ the associated stationary subdivision operator defined in* (1). *The operator is said to be C^0–*convergent *if for any data $\boldsymbol{g}_0 \in \ell^r(\mathbb{Z})$ and corresponding sequence of refinements*

$\boldsymbol{g}_n = S_{\boldsymbol{B}}^n \boldsymbol{g}_0$ *there exists a function* $\Psi_{\boldsymbol{g}} \in C(\mathbb{R}, \mathbb{R}^r)$ *such that for any compact* $K \subset \mathbb{R}$ *there exists a sequence* ε_n *with limit* 0 *that satisfies*

$$\max_{\alpha \in \mathbb{Z} \cap 2^n K} \left\| \boldsymbol{g}_n(\alpha) - \Psi_{\boldsymbol{g}}\left(2^{-n}\alpha\right) \right\|_\infty \leq \varepsilon_n. \tag{13}$$

Definition 7 *Let* $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *be a mask and* $H_{\boldsymbol{A}}$ *the associated Hermite subdivision scheme on* $\ell^{d+1}(\mathbb{Z})$ *as defined in* (2). *The scheme is called* convergent *if for any data* $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ *and the corresponding sequence of refinements* $\boldsymbol{f}_n$, *there exists a function* $\Phi = [\phi_i]_{i=0,\ldots,d} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ *such that for any compact* $K \subset \mathbb{R}$ *there exists a sequence* ε_n *with limit* 0 *which satisfies*

$$\max_{i=0,\ldots,d} \max_{\alpha \in \mathbb{Z} \cap 2^n K} \left| f_n^{(i)}(\alpha) - \phi_i\left(2^{-n}\alpha\right) \right| \leq \varepsilon_n. \tag{14}$$

The scheme $H_{\boldsymbol{A}}$ *is said to be* C^d–convergent *if moreover* $\phi_0 \in C^d(\mathbb{R}, \mathbb{R})$ *and*

$$\frac{d^i \phi_0}{dx^i} = \phi_i, \quad i = 0, \ldots, d.$$

Theorem 8 *Given a mask* $\boldsymbol{A} \in \ell^{d+1}(\mathbb{Z})$ *which satisfies the spectral condition. Suppose that for any data* $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ *and associated refinement sequence* $\boldsymbol{f}_n$ *of the Hermite scheme* $H_{\boldsymbol{A}}$,

1. *the sequence* $\boldsymbol{f}_n(0)$ *converges to a limit* $\boldsymbol{y} \in \mathbb{R}^{d+1}$,
2. *(at least) one of the following two properties holds true:*
   (a) the associated Taylor subdivision scheme $S_{\boldsymbol{B}}$ *is* C^0*–convergent and for any initial data* $\boldsymbol{g}_0 = T_d \boldsymbol{f}_0$, *the limit function* $\Psi = \Psi_{\boldsymbol{g}} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ *satisfies*
   $$\Psi = \begin{bmatrix} \mathbf{0} \\ \psi_d \end{bmatrix}, \; \psi_d \in C(\mathbb{R}, \mathbb{R}). \tag{15}$$
   (b) the associated complete Taylor subdivision scheme $S_{\widetilde{\boldsymbol{B}}}$ *is* contractive, *that is, it is* C^0*–convergent and for any initial data* $\boldsymbol{g}_0 = \widetilde{T}_d \boldsymbol{f}_0$, *the limit function* $\Psi = \mathbf{0} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$.

Then $H_{\boldsymbol{A}}$ *is convergent.*

The construction of the limit function Φ of $H_{\boldsymbol{A}}$ starts with a lemma in the special case $d = 1$.

Lemma 9 *Given a sequence of refinements*

$$\boldsymbol{h}_n = \begin{bmatrix} h_n^{(0)} \\ h_n^{(1)} \end{bmatrix} \in \ell^2(\mathbb{Z})$$

such that

1. *there exists a constant* c *in* $\mathbb{R}$ *such that* $\lim_{n\to+\infty} h_n^{(0)}(0) = c$,

2. *there exists a function $\xi \in C(\mathbb{R},\mathbb{R})$ such that for any compact subset K of $\mathbb{R}$ there exists a sequence μ_n with limit 0 and*

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left|h_n^{(1)}(\alpha) - \xi\left(2^{-n}\alpha\right)\right| \leq \mu_n, \tag{16}$$

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left|2^n \Delta h_n^{(0)}(\alpha) h_n^{(1)}(\alpha)\right| \leq \mu_n. \tag{17}$$

Then there exists for any compact K a sequence θ_n with limit 0 such that the function

$$\varphi(x) = c + \int_0^x \xi(t)dt, \qquad x\in\mathbb{R}, \tag{18}$$

satisfies

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left|h_n^{(0)}(\alpha) - \varphi\left(2^{-n}\alpha\right)\right| \leq \theta_n, \tag{19}$$

Proofs of Theorem 8 and Lemma 9 were given in [3] in the univariate case and a new proof which works for any number of variables was provided in [9].

Theorem 10 *Suppose that the mask $\boldsymbol{A}\in\ell^{d+1}(\mathbb{Z})$ satisfies the generalized spectral condition with a matrix $\boldsymbol{R}$ which is an upper triangular matrix with 1 on the diagonal. Let $\boldsymbol{B}_{\boldsymbol{R}}$ be the associated stationary subdivision operator such that*

$$T_d \boldsymbol{R}^{-1} S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}_R} T_d \boldsymbol{R}^{-1}.$$

Moreover, assume that

1. *for any data $\boldsymbol{f}_0\in\ell^{d+1}(\mathbb{Z})$ and associated refinement sequence $\boldsymbol{f}_n$ of the Hermite scheme $H_{\boldsymbol{A}}$, the sequence $\boldsymbol{f}_n(0)$ converges to a limit $\boldsymbol{y}\in\mathbb{R}^{d+1}$,*

2. *$S_{\boldsymbol{B}_R}$ is C^0–convergent and for any initial data $\boldsymbol{g}_0 = T_d\boldsymbol{f}_0$, the limit function $\Psi = \Psi_{\boldsymbol{g}} \in C\left(\mathbb{R},\mathbb{R}^{d+1}\right)$ satisfies*

$$\Psi = \begin{bmatrix}\boldsymbol{0}\\ \psi_d\end{bmatrix}, \ \psi_d\in C(\mathbb{R},\mathbb{R}). \tag{20}$$

Then $H_{\boldsymbol{A}}$ is C^d–convergent.

Before we prove the theorem, let us introduce a notation. For a compact $K\subset\mathbb{R}$ and a function $f\in C(K,\mathbb{R})$ we will write the modulus of continuity as

$$\omega(f,h) = \max_{x,y\in K,\,|x-y|\leq h} |f(x)-f(y)|.$$

Since K is compact, f is uniformly continuous and this value tends to 0 when h tends to 0.

Proof: The hypothesis $1.$ yields that for any $j<d$ the sequence $f_n^{(j)}(0)$ converges to y_j. Let us define Φ recursively as $\phi_d = \psi_d$, and for $j = d-1,\ldots,0$, as

$$\phi_j(x) = c_j + \int_0^1 x\phi_{j+1}(tx)dt. \tag{21}$$

Clearly, $\Phi = [\phi_j]_{0\le j\le d}$ is C^0 with $\phi_j' = \phi_{j+1}$ for $j = 0,\dots,d-1$. We will prove that the sequence $\boldsymbol{f}_n$ converges to Φ.

Since $\boldsymbol{R}$ is an upper triangular matrix with 1 on the diagonal, the matrix $\boldsymbol{R}^{-1}$, whose components we denote by r_{jk}, $j,k = 0,\dots,d$, has the same property and we obtain:

$$
\begin{aligned}
&T_d\boldsymbol{R}^{-1}\\
&\quad= \begin{bmatrix}
\Delta & -1 & -1/2! & -1/3! & \dots & -1/d!\\
0 & \Delta & -1 & -1/2! & \dots & -1/(d-1)!\\
0 & 0 & \Delta & -1 & \dots & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 0 & \Delta & -1\\
 & & & & 0 & 1
\end{bmatrix}\\
&\qquad\times\begin{bmatrix}
1 & r_{01} & r_{02} & r_{03} & \dots & r_{0d}\\
0 & 1 & r_{12} & r_{13} & \dots & r_{1d}\\
0 & 0 & 1 & r_{23} & \dots & r_{2d}\\
 & & \ddots & \ddots & & \\
 & & & 0 & 1 & r_{d-1d}\\
 & & & & 0 & 1
\end{bmatrix}\\
&\quad= \begin{bmatrix}
\Delta & -1+r_{01}\Delta & s_{02}+r_{02}\Delta & s_{03}+r_{03}\Delta & \dots & s_{0d}+r_{0d}\Delta\\
0 & \Delta & -1+r_{12}\Delta & s_{13}+r_{13}\Delta & \dots & s_{1d}+r_{1d}\Delta\\
0 & 0 & \Delta & -1+r_{23}\Delta & \dots & s_{2d}+r_{2d}\Delta\\
 & & \ddots & \ddots & \ddots & \\
 & & & 0 & \Delta & -1+r_{d-1d}\Delta\\
 & & & & 0 & 1
\end{bmatrix},
\end{aligned}
$$

where

$$s_{jk} = -\sum_{\ell=j+1}^{k} \frac{r_{\ell k}}{(\ell-j)!}, \qquad j,k = 0,\dots,d.$$

For initial data $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$, if we define $\boldsymbol{g}_n = 2^{nd}T_d\boldsymbol{R}^{-1}D^n\boldsymbol{f}_n$, we deduce for $n \in \mathbb{N}_0$ that

$$
\begin{aligned}
\boldsymbol{g}_{n+1} &= 2^{(n+1)d}T_d\boldsymbol{R}^{-1}D^{n+1}\boldsymbol{f}_{n+1} = 2^{(n+1)d}T_d\boldsymbol{R}^{-1}S_{\boldsymbol{A}}D^n\boldsymbol{f}_n\\
&= 2^{(n+1)d}2^{-d}S_{\boldsymbol{B}_R}T_d\boldsymbol{R}^{-1}D^n\boldsymbol{f}_n = S_{\boldsymbol{B}_R}\boldsymbol{g}_n,
\end{aligned}
$$

and so we obtain $\boldsymbol{g}_n = (S_{\boldsymbol{B}_R})^n \boldsymbol{g}_0$. The components of $\boldsymbol{g}_n$ can also be written as

$$
\begin{aligned}
g_n^{(0)} &= 2^{nd}\left(\Delta f_n^{(0)} - \frac{1}{2^n} f_n^{(1)} + r_{01}\frac{1}{2^n}\Delta f_n^{(1)} + \sum_{j=2}^{d}\frac{1}{2^{nj}}(s_{0j} + r_{0j}\Delta) f_n^{(j)}\right),\\
&\vdots\\
g_n^{(k)} &= 2^{nd}\left(\frac{1}{2^{nk}}\Delta f_n^{(k)} - \frac{1}{2^{n(k+1)}} f_n^{(k+1)}\right)\\
&\quad + 2^{nd}\left(r_{kk+1}\frac{1}{2^{n(k+1)}}\Delta f_n^{(k+1)} + \sum_{j=k+2}^{d}\frac{1}{2^{nj}}(s_{kj} + r_{kj}\Delta) f_n^{(j)}\right),\\
&\vdots\\
g_n^{(d-1)} &= 2^{nd}\left(\frac{1}{2^{n(d-1)}}\Delta f_n^{(d-1)} - \frac{1}{2^{nd}} f_n^{(d)} + r_{d-1,d}\frac{1}{2^{nd}}\Delta f_n^{(d)}\right),\\
g_n^{(d)} &= f_n^{(d)}.
\end{aligned}
$$

By assumption 2., the associated Taylor subdivision scheme $S_{\boldsymbol{B}_R}$ is convergent and the limit function $\Psi_{\boldsymbol{g}} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ satisfies (20). Fixing a compact $K \subset \mathbb{R}$, we then have that

$$
\begin{aligned}
\max_{\gamma\in 2^n K\cap\mathbb{Z}} \left|g_n^{(k)}(\gamma)\right| &\le \lambda_n, \qquad k = 0,\ldots,d-1,\\
\max_{\gamma\in 2^n K\cap\mathbb{Z}} \left|g_n^{(d)}(\gamma) - \psi_d(\gamma/2^n)\right| &\le \lambda_n,
\end{aligned}
\tag{22}
$$

with $\lambda_n \to 0$ as $n \to \infty$. Let us prove by a backward finite recursion for $k = d, d-1, \ldots, 0$ that

$$
\max_{\gamma\in 2^n K\cap\mathbb{Z}} \left|f_n^{(k)}(\gamma) - \phi_k(\gamma/2^n)\right| \le \mu_n^k, \qquad \lim_{n\to+\infty} \mu_n^k = 0. \tag{23}
$$

The case $k = d$ is an immediate consequence of the convergence of $g_n^{(d)} = f_n^{(d)}$, which yields for any $\gamma \in \mathbb{Z} \cap 2^n K$ that

$$
\left|f_n^{(d)}(\gamma) - \phi_d\left(2^{-n}\gamma\right)\right| \le \mu_n^d =: \lambda_n. \tag{24}
$$

In the case $k = d-1$, we obtain for $\gamma \in 2^n K \cap \mathbb{Z}$ that

$$
\left|2^n \Delta f_n^{(d-1)}(\gamma) - f_n^{(d)}(\gamma) + r_{d-1d}\Delta f_n^{(d)}(\gamma)\right| = |g_n^{d-1}(\gamma)| \le \lambda_n
$$

from which we deduce the following:

$$
\begin{aligned}
&\left|2^n \Delta f_n^{(d-1)}(\gamma) - f_n^{(d)}(\gamma)\right|\\
&\le \lambda_n + |r_{d-1,d}|\left(\left|f_n^{(d)}(\gamma+1) - \phi_d((\gamma+1)/2^n)\right|\right.\\
&\qquad \left. + |\phi_d((\gamma+1)/2^n) - \phi_d(\gamma/2^n)| + \left|\phi_d(\gamma/2^n) - f_n^{(d)}(\gamma)\right|\right)\\
&\le \lambda_n + |r_{d-1d}|\left(\mu_n^d + \omega(\phi_d, 1/n) + \mu_n^d\right).
\end{aligned}
$$

Hence,

$$\lim_{n\to+\infty} \max_{\gamma\in 2^nK\cap\mathbb{Z}} \left| 2^n \Delta f_n^{(d-1)}(\gamma) - f_n^{(d)}(\gamma) \right| = 0. \tag{25}$$

To prove (23) for $k = d-1$, we define the sequences $h_n^{(1)} = f_n^{(d)}$, $h_n^{(0)} = f_n^{(d-1)}$ as well as $\xi = \phi_d$. Because of (24) and (25), we can apply Lemma 9 and we obtain that

$$\left| f_n^{(d-1)}(\gamma) - \phi_{d-1}\left(2^{-n}\gamma\right) \right| \le \mu_n^{d-1}, \qquad \gamma \in 2^nK\cap\mathbb{Z},$$

which is (23) for $k = d-1$.

To prove the recursive step $k+1 \to k$ for $0 \le k < d-1$, we first notice that for $j = k+1, \ldots, d$, the functions ϕ_j are continuous on K, so that they are bounded. The sequences $f_n^{(j)}(\gamma)$ are also uniformly bounded for $\gamma \in 2^nK\cap\mathbb{Z}$ since $\left| f_n^{(j)}(\gamma) - \phi_j\left(2^{-n}\gamma\right) \right| \le \mu_n^j$ with $\lim \mu_n^j = 0$. Let us write M_j for this common bound. We also notice that for for $\gamma \in 2^nK\cap\mathbb{Z}$,

$$\begin{aligned} \left|\Delta f_n^{(k+1)}(\gamma)\right| &\le \left| f_n^{(k+1)}(\gamma+1) - \phi_{k+1}((\gamma+1)/2^n) \right| \\ &\quad + \left|\phi_{k+1}((\gamma+1)/2^n) - \phi_{k+1}(\gamma/2^n)\right| \\ &\quad + \left|\phi_{k+1}(\gamma/2^n) - f_n^{(k+1)}(\gamma)\right| \\ &\le 2\mu_n^{k+1} + \omega(\phi_{k+1}, 1/n) \end{aligned}$$

and for $j = k+2, \ldots, d$, we have $\left|\Delta f_n^{(k+1)}(\gamma)\right| \le 2M_k$.

For $\gamma \in \mathbb{Z}\cap 2^nK$, when multiplying $|g_n^k(\gamma)| \le \lambda_n$ by $2^{n(k+1-d)}$, we obtain by means of (22) that

$$\begin{aligned} &\left| 2^n\Delta f_n^{(k)}(\gamma) - f_n^{(k+1)}(\gamma) \right| \\ &\le \lambda_n 2^{n(k+1-d)} + \left| r_{kk+1}\Delta f_n^{(k+1)}(\gamma) \right| + \sum_{j=k+2}^{d} \frac{1}{2^{n(j-k-1)}} \left| (s_{kj} + t_{kj}\Delta) f_n^{(j)}(\gamma) \right| \\ &\le \lambda_n 2^{n(k+1-d)} + |r_{kk+1}| \left(2\mu_n^{k+1} + \omega(\phi_{k+1}, 1/n) \right) + \sum_{j=k+2}^{d} \frac{(|s_{kj}| + 2|t_{kj}|)M_k}{2^{n(j-k-1)}}. \end{aligned}$$

Since $k+1-d < 0$,

$$\lim_{n\to+\infty} \max_{\gamma\in 2^nK\cap\mathbb{Z}} \left| 2^n \Delta f_n^{(k)}(\gamma) - f_n^{(k+1)}(\gamma) \right| = 0. \tag{26}$$

The rest of the proof proceeds as in the special case $k = d-1$ above. □

6 Examples

We finally illustrate the concept of generalized Taylor factorizations by looking at some examples of cardinal spline schemes in the spirit of Section 3.2.

6.1 Cardinal Spline with $r = 4$, $d = 2$

The coefficients in the first column are given according to the following table:

α	0	1	2	3	4	5	6	7
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1

With $p_0(x) = 1$, $p_1(x) = x + \frac{5}{2}$, $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$ and

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain that $S_{\boldsymbol{A}}\boldsymbol{R}\boldsymbol{v}_j = 2^{-j}\boldsymbol{R}\boldsymbol{v}_j$ where $\boldsymbol{v}_j = \begin{bmatrix} p_j \\ p_j' \\ p_j'' \end{bmatrix}$, for $j = 0, 1, 2$

6.2 Cardinal spline with $r = 4$, $d = 3$

α	0	1	2	3	4	5	6	7	8
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1	0
$16a_{30}(\alpha)$	1	2	-2	-6	0	6	2	-2	-1

Here we have $p_0(x) = 1$, $p_1(x) = x + \frac{5}{2}$, $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$, $p_3(x) = \frac{1}{6}x^3 + \frac{5}{4}x^2 + \frac{35}{12}x + \frac{25}{12}$ and mention that it is worthwhile to notice that $p_i' = p_{i-1}$. Now the choice

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

leads to $S_{\boldsymbol{A}}\boldsymbol{R}\boldsymbol{v}_j = 2^{-j}\boldsymbol{R}\boldsymbol{v}_j$. For $\boldsymbol{A}_R = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$ some straightforward computations yield the *symbol representation*, cf.[9],

$$\boldsymbol{A}_R^*(z) = \frac{1}{96}\begin{bmatrix} 6(1+z)^5 & 0 & 0 & 0 \\ (1+z)^5(1-z)(11-7z+2z^2) & 0 & 0 & 0 \\ 6(1+z)^5(1-z)^2(2-z) & 0 & 0 & 0 \\ 6(1+z)^5(1-z)^3 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain that $\widetilde{\boldsymbol{B}}_R$ defined by $\widetilde{\boldsymbol{T}}\boldsymbol{A}_R = 2^{-3}\widetilde{\boldsymbol{B}}_R\widetilde{\boldsymbol{T}}_d$ has the coefficients $\widetilde{\boldsymbol{B}}_R(0) = \mathbf{0}$ and

$$
\begin{aligned}
\widetilde{\boldsymbol{B}}_R(1) &= \frac{1}{12}\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{11}{6} & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \widetilde{\boldsymbol{B}}_R(2), \\
\widetilde{\boldsymbol{B}}_R(3) &= \frac{1}{12}\begin{bmatrix} -3 & 1 & \frac{1}{2} & \frac{1}{6} \\ -\frac{11}{2} & \frac{11}{6} & \frac{11}{12} & \frac{11}{36} \\ -6 & 2 & 1 & \frac{1}{3} \\ -3 & 1 & \frac{1}{2} & \frac{1}{6} \end{bmatrix} = \widetilde{\boldsymbol{B}}_R(4), \\
\widetilde{\boldsymbol{B}}_R(5) &= \frac{1}{12}\begin{bmatrix} 3 & -2 & 0 & \frac{2}{3} \\ \frac{11}{2} & -\frac{11}{3} & 0 & \frac{11}{9} \\ 6 & -4 & 0 & \frac{4}{3} \\ 3 & -2 & 0 & \frac{2}{3} \end{bmatrix} = \widetilde{\boldsymbol{B}}_R(6), \\
\widetilde{\boldsymbol{B}}_R(7) &= \frac{1}{12}\begin{bmatrix} -1 & 1 & -\frac{1}{2} & \frac{1}{6} \\ -\frac{11}{6} & \frac{11}{6} & -\frac{11}{12} & \frac{11}{36} \\ -2 & 2 & -1 & \frac{1}{3} \\ -1 & 1 & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} = \widetilde{\boldsymbol{B}}_R(8).
\end{aligned}
$$

Employing the matrix

$$
\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11/6 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
$$

and the norm $\|x\| = \|\boldsymbol{P}^{-1}x\boldsymbol{P}\|_1$, we have that

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{B}}_R(1)\| = \|\widetilde{\boldsymbol{B}}_R(2)\| = 1/12, \quad \|\widetilde{\boldsymbol{B}}_R(3)\| = \|\widetilde{\boldsymbol{B}}_R(4)\| = 1/12, \\
&\|\widetilde{\boldsymbol{B}}_R(5)\| = \|\widetilde{\boldsymbol{B}}_R(6)\| = 2/12, \quad \|\widetilde{\boldsymbol{B}}_R(7)\| = \|\widetilde{\boldsymbol{B}}_R(8)\| = 1/12.
\end{aligned}
$$

Thus $\sum_{\alpha\in\mathbb{Z}} \|\widetilde{\boldsymbol{B}}_R(2\alpha)\| = \sum_{\alpha\in\mathbb{Z}} \|\widetilde{\boldsymbol{B}}_R(2\alpha+1)\| = 5/12 < 1$ and therefore the operator $S_{\widetilde{\boldsymbol{B}}_R}$ is contractive, re-proving the known result that the Hermite scheme is C^4.

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