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Extra Regularity of Hermite Subdivision Schemes

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Abstract

Hermite subdivision schemes act on vector valued sequences that are not only considered as functions values of a vector valued function from $\mathbb{R}$ to $\mathbb{R}^r$, but as evaluations of a function and its consecutive derivatives. Starting with data on $\ell^r(\mathbb{Z})$, $r = d+1$, interpreted as function value and $d = r-1$ consecutive derivatives, we compute successive iterations to define values on $\ell^r(2^{-n}\mathbb{Z})$ and an r-vector valued limit function for whose first component C^d–smoothness is generally expected.

In this paper, we construct univariate Hermite subdivision schemes such that, for any given initial data, it is possible to reach a limit function with smoothness $d + p$ for any $p > 0$. The result is obtained with a generalized Taylor factorization and a smoothness condition for vector subdivision schemes.

1 Introduction

Subdivision schemes create curves or surfaces by applying stationary refinement rules on data defined on the integers. This refinement process extends the data to a discrete function defined on the half integers, quarter integers and so on, until eventually the values become so dense that one could speak of a *limit function*. In the univariate case, which is the one we consider here, *stationary subdivision* [1] means that any step of the subdivision process is a stationary process which defines data on next level in a convolution-like way as

$$g_{n+1} = S_a g_n := \sum_{\beta\in\mathbb{Z}} a(\cdot - 2\beta)\, g_n(\beta);$$

in this expression, a stands for the *mask*, a finitely supported sequence and the values g_n on different iteration levels are normalized to be discrete functions $g_n : \mathbb{Z} \to \mathbb{R}$ with the understanding that $g_n(\alpha)$ stands for a value at $2^{-n}\alpha$, $\alpha \in \mathbb{Z}$. Such subdivision schemes with scalar coefficients can be trivially extended to the generation of curves by acting *componentwise*, on vector data, resulting in the iteration

$$\boldsymbol{g}_{n+1} = S_a \boldsymbol{g}_n := \sum_{\beta\in\mathbb{Z}} a(\cdot - 2\beta)\, \boldsymbol{g}_n(\beta), \qquad \boldsymbol{g}_n : \mathbb{Z} \to \mathbb{R}^r.$$

Vector subdivision goes one step further by applying a *matrix valued* mask to the data, allowing for interaction between the components of the data vectors:

$$\boldsymbol{g}_{n+1} = S_{\boldsymbol{A}}\, \boldsymbol{g}_n := \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\cdot - 2\beta)\, \boldsymbol{g}_n(\beta), \qquad \boldsymbol{A} : \mathbb{Z} \to \mathbb{R}^{r\times r},$$

again with the assumption that $\boldsymbol{A}$ is finitely supported. Finally, in *Hermite subdivision* the components of the vector $\boldsymbol{f}_n(\alpha) \in \mathbb{R}^r$, $r = d + 1$, are considered as function value and d consecutive derivatives of a function at $2^{-n}\alpha$. Due to the chain rule, the refinement scheme now takes a *level dependent* form, that is, the operator depends on the iteration level n as

$$\boldsymbol{f}_{n+1} = \boldsymbol{D}^{-n-1} S_{\boldsymbol{A}} \boldsymbol{D}^n \boldsymbol{f}_n = \sum_{\beta\in\mathbb{Z}} \boldsymbol{D}^{-n-1} \boldsymbol{A}(\cdot - 2\beta) \boldsymbol{D}^n \boldsymbol{f}_n(\beta), \qquad \boldsymbol{D} = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & 2^{-d} \end{bmatrix}.$$

All such types of subdivision schemes are covered extensively in the literature, see e.g. [2, 3, 4, 5, 8, 15], just to name a few specific references on Hermite subdivision schemes. Standard questions to consider are the convergence of the iterative schemes and the regularity of the associated limit functions. This is well-known to be closely related to the way how the subdivision operators act on polynomial sequences, a property that can in turn be conveniently characterized by means of operator factorizations.

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In the next section, we will review the basic definitions of vector and Hermite subdivision schemes and the appropriate notions of convergence. We will point out what vector subdivision schemes and Hermite subdivision schemes have in common and where they differ. Introducing Taylor operators, we will also present the transformation of a Hermite subdivision scheme into vector subdivision schemes via the Taylor factorizations. We illustrate the different schemes with an example where, in particular the limit functions are shown.

We will see that the definitions of the smoothness of the two schemes are significantly different. By construction, the *limit function*

$$\phi = \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_d \end{bmatrix},$$

of a Hermite scheme satisfies $\phi_j = \phi_0^{(j)}$ for $j = 0, \dots, d$, so that $\phi_0 \in C^d$ whenever all components of ϕ are continuous. The limit of a Hermite subdivision scheme always has to have a certain amount of regularity in the sense of differentiability. In this paper we investigate the question under which circumstances we can have extra regularity, that is, $\phi_0 \in C^{d+p}$ for some integer $p \geq 0$. We will relate this to a combined factorization, one due to the nature of Hermite subdivision schemes and one coming from a smoothness condition for vector subdivision schemes that is due to [14]. A similar approach has been used to characterize *overreproduction* of polynomials as an algebraic properties of the matrix symbols in [16, 17].

Section 3 will be devoted to the B-spline case. Here the splines are obtained as the limit of, firstly, a scalar subdivision scheme, secondly, a Hermite subdivision scheme. The smoothness of such functions is well-known and can be as large as wanted.

In the final Section 4, we will give a generic construction to obtain convergent Hermite subdivision schemes with any order of extra smoothness.

2 Vector and Hermite subdivision schemes

We begin by fixing some notation to describe subdivision schemes. Vectors in $\mathbb{R}^r$, $r \in \mathbb{N}$, will generally be labeled by lowercase boldface letters: $\boldsymbol{y} = \left[y_j\right]_{j=0,\dots,r-1}$ or $\boldsymbol{y} = \left[y^{(j)}\right]_{j=0,\dots,r-1}$, where the latter notation is used whenever we especially want to highlight the aforementioned fact that in Hermite subdivision the components of the vectors correspond to consecutive derivatives. Moreover, in Hermite subdivision we denote the highest derivative by d, so that throughout the paper we will always have the relationship $r = d + 1$.

Matrices in $\mathbb{R}^{r\times r}$ will be written as uppercase boldface letters such as $\boldsymbol{A} = \left[a_{jk}\right]_{j,k=0,\dots,r-1}$. The space of polynomials in one variable of degree at most n will be written as Π_n, with the usual convention $\Pi_{-1} = \{0\}$, while Π will denote the space of all polynomials. Vector sequences will be considered as functions from $\mathbb{Z}$ to $\mathbb{R}^r$ and the vector space of all such functions will be denoted by $\ell^r(\mathbb{Z})$. For a sequence $\boldsymbol{y} \in \ell^r(\mathbb{Z})$, the *forward difference* is defined as $\Delta \boldsymbol{y} := \boldsymbol{y}(\cdot+1) - \boldsymbol{y}$, and iterated to

$$\Delta^j \boldsymbol{y} := \Delta\left(\Delta^{j-1}\boldsymbol{y}\right) = \Delta^{j-1}\boldsymbol{y}(\cdot+1) - \Delta^{j-1}\boldsymbol{y}(\cdot) = \sum_{k=0}^{j} (-1)^{k-j}\binom{j}{k}\boldsymbol{y}(\cdot+k), \qquad j \geq 1.$$

We use $\mathbf{0}$ to indicate zero vectors and matrices. If we want to highlight the dimension of the object, we will use a subscript like $\mathbf{0}_r$, but to avoid too cluttered notation, we will often drop them if the size of the object is clear from the context.

For a finitely supported sequence of matrices $\boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z})$, called the *mask* of the subdivision scheme, we define the associated *stationary subdivision operator*

$$S_{\boldsymbol{A}} : \boldsymbol{g} \mapsto \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\cdot - 2\beta)\,\boldsymbol{g}(\beta), \qquad \boldsymbol{g} \in \ell^r(\mathbb{Z}).$$

Using potentially different masks $\boldsymbol{A}_n \in \ell_0^{r\times r}(\mathbb{Z})$, $n \in \mathbb{N}$, these operators can be iterated into a *subdivision scheme* that creates sequences $\boldsymbol{g}_n \in \ell_0^r(\mathbb{Z})$, $n \geq 0$,

$$\boldsymbol{g}_{n+1} := S_{\boldsymbol{A}_n}\boldsymbol{g}_n := \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}_n(\cdot - 2\beta)\,\boldsymbol{g}_n(\beta), \qquad n \geq 0, \tag{1}$$

from a given $\boldsymbol{g}_0$. An important algebraic tool for stationary subdivision operators is the *symbol* of the mask, the matrix valued Laurent polynomial

$$\boldsymbol{A}^*(z) := \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}(\alpha)\,z^\alpha, \qquad z \in \mathbb{C}\setminus\{0\}. \tag{2}$$

In a *vector subdivision scheme* as defined in [1], we simply set $\boldsymbol{A}_n = \boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z})$ and define convergence as follows.

Definition 2.1. The *vector subdivision operator* $S_{\boldsymbol{A}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$ is called C^p*–convergent*, $p \geq 0$, if for any data $\boldsymbol{g}_0 := \boldsymbol{g} \in \ell^r(\mathbb{Z})$ and the refinements from (1) there exists a function $\psi_{\boldsymbol{g}} : \mathbb{R} \to \mathbb{R}^r$ with C^p components such that for any compact $K \subset \mathbb{R}$ there exists a sequence ε_n with limit 0 that satisfies

$$\max_{\alpha\in\mathbb{Z}\cap 2^n K} \left\|\boldsymbol{g}_n(\alpha) - \psi_{\boldsymbol{g}}\left(2^{-n}\alpha\right)\right\|_\infty \leq \varepsilon_n. \tag{3}$$

For a *Hermite scheme*, in (1), we set

$$\boldsymbol{A}_n(\alpha) = \boldsymbol{D}^{-n-1}\boldsymbol{A}(\alpha)\boldsymbol{D}^n, \qquad \alpha \in \mathbb{Z}, \qquad \boldsymbol{D} := \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^d} \end{bmatrix}, \tag{4}$$

so that $r = d+1$ and for $k = 0,\dots,d$ the k-th component of $\boldsymbol{c}_n(\alpha)$ corresponds to an approximation of the k-th derivative of some function φ_n at $\alpha 2^{-n}$. Starting from an initial sequence $\boldsymbol{f}_0 \in \ell^r(\mathbb{Z})$, a Hermite scheme

$$\boldsymbol{f}_{n+1} := H_{A_n}\boldsymbol{f}_n := \boldsymbol{D}^{-n-1} S_A \boldsymbol{D}^n \boldsymbol{f}_n, \qquad n \geq 0,$$

can be rewritten as

$$\boldsymbol{g}_{n+1} := \boldsymbol{D}^{n+1}\boldsymbol{f}_{n+1} = S_A \boldsymbol{D}^n \boldsymbol{f}_n = S_A \boldsymbol{g}_n, \qquad n \geq 0, \tag{5}$$

based on the relation

$$\boldsymbol{g}_n = \boldsymbol{D}^n \boldsymbol{f}_n, \qquad n \geq 0. \tag{6}$$

To capture the intuition of vectors with consecutive derivatives, the convergence of Hermite schemes is a little bit more intricate and defined as follows.

Definition 2.2. The Hermite subdivision scheme with respect to the mask $\boldsymbol{A} \in \ell^{r\times r}(\mathbb{Z})$ as defined by (5) is called *convergent* if for any data $\boldsymbol{f}_0 \in \ell^r(\mathbb{Z})$ there exists a function $\Phi : \mathbb{R} \to \mathbb{R}^r$ with (uniformly) continuous components such that for any compact $K \subset \mathbb{R}$ there exists a sequence ε_n with limit 0 which satisfies

$$\max_{0\leq j\leq d} \max_{\alpha\in\mathbb{Z}\cap 2^n K} \left| f_n^{(j)}(\alpha) - \phi_i\left(2^{-n}\alpha\right)\right| \leq \varepsilon_n. \tag{7}$$

Moreover, the scheme H_{A_n} is said to be C^p*–convergent* with $p \geq d$ if in addition ϕ_0 is a C^p function and

$$\phi_0^{(j)} = \phi_j, \qquad 0 \leq j \leq d.$$

Remark 1. Since the intuition of Hermite subdivision schemes is to iterate on function values and derivatives, it usually only makes sense to consider C^p–convergence for $p \geq d$. Note, however, that the case $p > d$ leads to additional requirements.

Remark 2. The two concepts of convergence based on S_A and H_{A_n} are significantly different as can be seen immediately from (6). Indeed, if the Hermite subdivision scheme is convergent it follows that

$$\boldsymbol{g}_n = \begin{bmatrix} f_n^{(0)} \\ 2^{-n} f_n^{(1)} \\ \vdots \\ 2^{-nd} f_n^{(d)} \end{bmatrix} \to \begin{bmatrix} \phi_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

hence $\Psi_g = \phi_0 \boldsymbol{e}_0$. In particular, the components of $\boldsymbol{g}_n$ have to converge to zero even with a prescribed rate. Therefore, in general it cannot be ensured that $\boldsymbol{D}^{-n}\boldsymbol{g}_n$ converges or is bounded, even if $\boldsymbol{g}_n$ converges to a multiple of $\boldsymbol{e}_0$. This is the reason why the factorization properties and the convergence analysis for Hermite subdivision cannot be obtained in a straightforward way from that of the vector subdivision operator, even if they are based on the same mask, see Fig. 1 for a particular example.

As a consequence of Remark 2 we observe that whenever a mask $\boldsymbol{A}$ defines a convergent Hermite subdivision scheme, the associated vector subdivision scheme based on S_A is a so-called *rank*-1 *subdivision scheme* as defined in [13, 14]. In concrete terms, this means that the mask as to satisfy

$$\boldsymbol{Q}_0\boldsymbol{e}_0 = \boldsymbol{Q}_1\boldsymbol{e}_0 = \boldsymbol{e}_0, \qquad \boldsymbol{Q}_\epsilon := \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}_{2\alpha+1}, \quad \epsilon \in \{0,1\},$$

and that the two matrices $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ have no further common eigenvector with respect to the eigenvalue 1.
To give convergence criteria for vector and Hermite subdivision schemes, we need three different types of difference operators from [10, 11].

Definition 2.3. The *simple difference operator* (of rank-1 type) is defined as

$$D_d := I + (\Delta - 1)\boldsymbol{e}_d\boldsymbol{e}_d^T = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \Delta \end{bmatrix}. \tag{8}$$

A *generalized incomplete Taylor operator* is an operator of the form

$$T_d := \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} \Delta I & \\ & 1 \end{bmatrix} + \begin{bmatrix} t_{jk} \end{bmatrix}_{j,k=0,\dots,d}, \tag{9}$$

where

$$t_{j,j+1} = -1 \qquad \text{and} \qquad t_{jk} = 0, \quad k \leq j.$$

In the same way, the *generalized complete Taylor operator* is of the form

$$\widetilde{T}_d := D_d\, T_d = \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix} = \Delta I + \left[t_{jk} \right]_{j,k=0,\dots,d}. \tag{10}$$

In this paper we only consider generalized Taylor operators, so that from now on we will drop the word "generalized" and always speak of generalized operators and factorizations.

The name *Taylor operator* stems from the fact that, motivated by observations from [6], the (incomplete) operator had originally be introduced in [10] as

$$T_d := \begin{bmatrix} \Delta & -1 & -\frac{1}{2} & \dots & -\frac{1}{d!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -\frac{1}{2} \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix}.$$

It is a particular case of our new generalized Taylor operator and it appears as a comparison between the difference of function values and the derivative terms of the Taylor expansion. Since for any $\phi \in C^{d+1}(\mathbb{R})$ one has that

$$\widetilde{T}_d \begin{bmatrix} \phi \\ \phi' \\ \vdots \\ \phi^{(d)} \end{bmatrix} (x) = \begin{bmatrix} \phi^{(d+1)}(\xi_0) \\ \phi^{(d+1)}(\xi_1) \\ \vdots \\ \phi^{(d+1)}(\xi_d) \end{bmatrix}, \qquad \xi_j \in (x, x+1), \quad j = 0, \dots, d,$$

the operator clearly annihilates all polynomials of degree at most d.

The generalized Taylor operatorenables us to give a sufficient criterion for the convergence of Hermite subdivision schemes by means of *factorization*.

Definition 2.4. The masks $\boldsymbol{B}, \widetilde{\boldsymbol{B}} \in \ell^{r\times r}(\mathbb{Z})$ are called a *Taylor factorization* and a *complete Taylor factorization* of $\boldsymbol{A}$, respectively, if they satisfy

$$T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d \qquad \text{and} \qquad \widetilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}} \widetilde{T}_d, \tag{11}$$

respectively.

In a certain sense, the factorization always exists. We simply have to write (11) as

$$\boldsymbol{T}^*(z)\boldsymbol{A}^*(z) = 2^{-d}\boldsymbol{B}^*(z)\boldsymbol{T}^*(z^2), \qquad \boldsymbol{T}^*(z) := \begin{bmatrix} z^{-1}-1 & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & z^{-1}-1 & -1 \\ & & & & 1 \end{bmatrix}$$

with an analogous identity for the complete factorization, to obtain that

$$\boldsymbol{B}^*(z) = 2^d \boldsymbol{T}^*(z)\boldsymbol{A}^*(z)\boldsymbol{T}^*(z^2)^{-1} \qquad \text{and} \qquad \boldsymbol{A}^*(z) = 2^{-d}\, \boldsymbol{T}^*(z)^{-1}\boldsymbol{B}^*(z)\boldsymbol{T}^*(z^2),$$

respectively. Given a finitely supported mask $\boldsymbol{A}$, the resulting $\boldsymbol{B}^*(z)$ is usually a non-polynomial rational function, hence the factor $\boldsymbol{B}$ is an infinitely supported mask. Unfortunately, the same also holds true for $\boldsymbol{A}^*$ which, for given $\boldsymbol{B}$ can only be guaranteed to be a rational function, even if

$$\begin{aligned} \det \boldsymbol{A}^*(z) &= 2^{-d} \left(\det \boldsymbol{T}^*(z)\right)^{-1} \det \boldsymbol{B}^*(z) \det \boldsymbol{T}^*(z^2) = 2^{-d} \frac{(z^{-2}-1)^{d+1}}{(z^{-1}-1)^{d+1}} \det \boldsymbol{B}^*(z) \\ &= 2 \left(\frac{z^{-1}+1}{2} \right)^{d+1} \det \boldsymbol{B}^*(z) \end{aligned}$$

is a Laurent polynomial in z. This is in contrast to scalar subdivision schemes where raising the order of the zero at -1 is the standard way to increase the smoothness of the limit function. Nevertheless, the existence of a factorization is the key to the construction of convergent Hermite subdivision schemes.

Theorem 2.1 ([11], Corollary 4). *If a given mask $\boldsymbol{A}$ has a complete Taylor factorization $\widetilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}} \widetilde{T}_d$ where*

1. $\widetilde{\boldsymbol{B}} \in \ell^{r\times r}(\mathbb{Z})$ *is* finitely supported,
2. $S_{\widetilde{\boldsymbol{B}}}$ *is a contraction on* $\ell^r_\infty(\mathbb{Z})$*,*

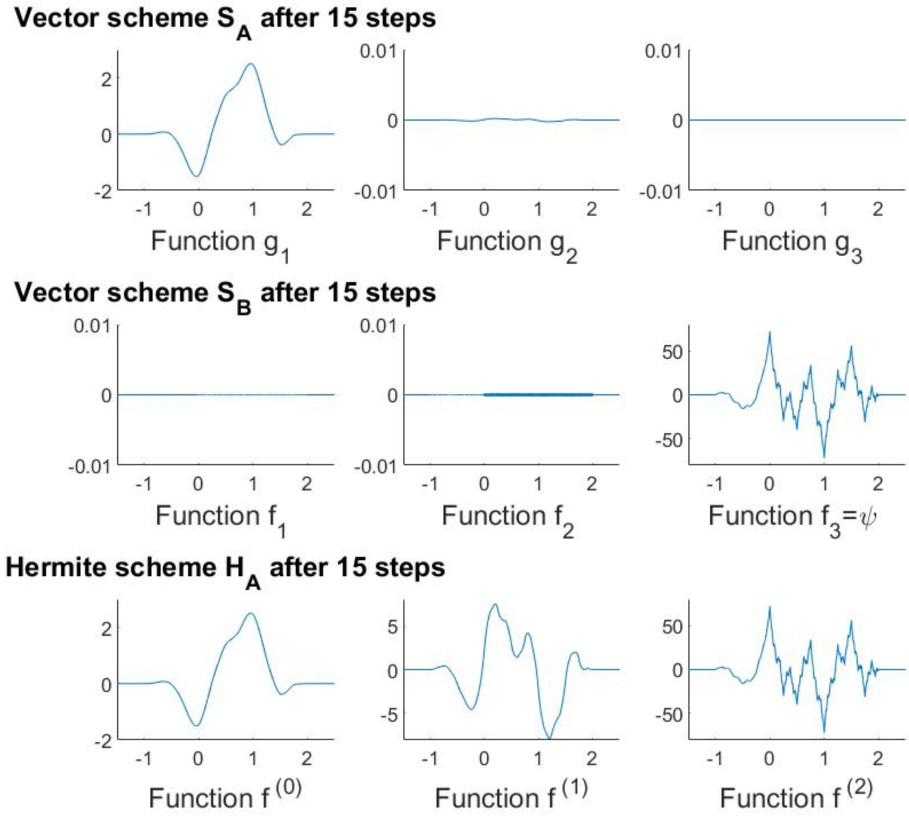

Figure 1: The different schemes: S_A, S_B and H_{A_n}

3. $(\boldsymbol{B}^*(1))_{11} = 1$,

then H_{A_n} is C^d-convergent.

Hence, in order to construct a C^d-convergent subdivision scheme, we can start with a finitely supported $\widetilde{\boldsymbol{B}}$ such that the associated subdivision satisfies the contractivity condition 2) and the normalization condition 3) at the same time. Note that the latter prohibits a simple rescaling of $\boldsymbol{B}$, i.e., a multiplication with a small constant.

This, however, is not enough as one also has to ensure that $\boldsymbol{T}^*(z)^{-1}\boldsymbol{B}^*(z)\boldsymbol{T}^*(z^2)$ is a matrix valued Laurent polynomial which leads to additional conditions on $\boldsymbol{B}^*$. A generic construction for such a $\boldsymbol{B}$ has been given, for *any* generalized Taylor operator, in [11] which shows that for any generalized Taylor operator and any d there exists a C^d convergent Hermite subdivision scheme that is factorizable with respect to this generalized Taylor operator. We will later extend this construction by means of a *supercomplete* Taylor factorization, but first we illustrate the concept by looking at a special case that actually motivated the development of generalized Taylor factorizations.

In Figure 1, with a given mask, $\{\boldsymbol{A}(\cdot)\}$, we plot the "limit" functions for the two vector schemes, S_A, S_B (after Taylor incomplete factorization), and the Hermite scheme H_{A_n}. We notice that the first functions for S_A and H_{A_n} are identical, corresponding to ϕ_0 in Definition 2.2 and similarly the last ones of S_B and H_{A_n} corresponding to $\phi_0^{(d)}$ in the same definition.

3 The B–spline case

In this section, we rewrite the well known cardinal splines, [18] in term of a scalar subdivision scheme and extend it into Hermite schemes of different orders. From the properties of cardinal splines, we have convergence of the schemes and regularity of the limit.

Our presentation, already proposed in [9], is based on a construction detailed by Michelli in [12] and in summarized in the following.

Let

$$\varphi_0(x) = \chi_{[0,1]} = \begin{cases} 1 \text{ if } x \in [0,1], \\ 0 \text{ if } x \notin [0,1]. \end{cases}$$

For $m = 1, 2, \ldots$, we build φ_m by means of autoconvolution as $\varphi_m = \varphi_0 * \varphi_{m-1}$ or $\varphi_m(x) = \int_{x-1}^{x} \varphi_{m-1}(t)dt$.

Let us recall that φ_m is a C^{m-1} piecewise polynomial of degree m with finite support $[0, m+1]$.

Moreover, $\varphi_m(x) = \frac{1}{2^m}\sum_{\alpha\in\mathbb{Z}}\binom{m+1}{\alpha}\varphi_m(2x-\alpha)$ where $\binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & \text{if } 0 \le j \le i, \\ 0 & \text{otherwise.} \end{cases}$

Considering $v(x) = \sum_{\alpha\in\mathbb{Z}} f_0^{(0)}(\alpha)\varphi_m(x-\alpha)$, which is a finite sum for any $x \in \mathbb{R}$ since φ_m has finite support, we deduce for $n \in \mathbb{N}_0$ that $v(x) = \sum_{\alpha\in\mathbb{Z}} f_n^{(0)}(\alpha)\varphi_m(2^n x - \alpha)$ where

$$f_{n+1}^{(0)}(\alpha) = \frac{1}{2^m}\sum_{\beta\in\mathbb{Z}}\binom{m+1}{\alpha-2\beta} f_n^{(0)}(\beta) =: \sum_{\beta\in\mathbb{Z}} a_m(\alpha-2\beta) f_n^{(0)}(\beta), \qquad \alpha\in\mathbb{Z}, \tag{12}$$

that is,

$$a_m(\alpha) = \frac{1}{2^m}\binom{m+1}{\alpha}, \qquad \alpha\in\mathbb{Z}. \tag{13}$$

This is a scalar subdivision scheme.

Then, the well–known derivative formula for cardinal B–spline yields

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} 2^{ni}\Delta^i f_n^{(0)}(\alpha-i)\,\varphi_{m-i}(2^n x-\alpha), \qquad i = 0,\dots,m-1. \tag{14}$$

We have a particular case when $i = m-1$. Since the function φ_1 is piecewise linear with $\varphi_1(\alpha) = \delta_{1\alpha}$, we obtain

$$\frac{d^{m-1}v}{dx^{m-1}}(\beta/2^n) = 2^{n(m-1)}\Delta^{m-1} f_n^{(0)}(\beta-m+1).$$

With this formula, we define a Hermite subdivision scheme of order $d < m$ with mask $\{\boldsymbol{A}(\alpha)\}$ and support $[0, m+d+1]$ by applying differences to the mask a_m, yielding

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} a_m(\alpha) & 0 & \dots & 0 \\ \Delta a_m(\alpha-1) & 0 & \dots & 0 \\ \vdots & & & \\ \Delta^d a_m(\alpha-d) & 0 & \dots & 0 \end{bmatrix}, \tag{15}$$

thus

$$\boldsymbol{A}^*(z) = \frac{(1+z)^{m+1}}{2^m}\begin{bmatrix} 1 & 0 & \dots & 0 \\ (1-z) & 0 & \dots & 0 \\ \vdots & & & \\ (1-z)^d & 0 & \dots & 0 \end{bmatrix}. \tag{16}$$

Beginning with $\boldsymbol{f}_0 \in \ell^r$, and using the recurrence (5) we notice that for $n \ge 1$ and $i = 1,\dots,d$:

$$f_n^{(i)}(\alpha) = 2^{in}\Delta^i f_n^{(0)}(\alpha-i). \tag{17}$$

Now with (12) and (14), for $n > 0$,

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} f_n^{(i)}(\alpha)\varphi_{m-i}(2^n x-\alpha), \qquad i = 0,\dots,d.$$

In [9], we had proved that the generalized Taylor operators are given by

$$\boldsymbol{T}_d := \begin{bmatrix} \Delta & -1 & \dots & -1 \\ & \ddots & \ddots & \vdots \\ & & \Delta & -1 \\ & & & 1 \end{bmatrix} \text{ and } \widetilde{\boldsymbol{T}}_d := \begin{bmatrix} \Delta & -1 & \dots & -1 \\ & \ddots & \ddots & \vdots \\ & & \Delta & -1 \\ & & & \Delta \end{bmatrix} \tag{18}$$

Thus the corresponding vector scheme in the factorization is given by

$$\widetilde{\boldsymbol{B}}^*(z) = \frac{z(1+z)^{m-d}}{2^{m-d}}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\begin{bmatrix} (1-z^2)^d & z^2((1-z^2)^{d-1} & \dots & z^2(1-z^2) & z^2 \end{bmatrix}. \tag{19}$$

It is of rank 1. Let us also notice that

$$\widetilde{\boldsymbol{T}}^*(z)\boldsymbol{A}^*(z) = 2^{-d}\widetilde{\boldsymbol{B}}^*(z)\boldsymbol{T}^*(z^2) = 2^{-m}z^{-1}(1+z)^{m+1}(1-z)^{d+1}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}.$$

We did not plot the graphs of the B-splines which are well known and probably everyone has seen one already.

4 Hermite schemes with extra regularity: a generic construction

In this section we will show that for any generalized Taylor operator of any order d there exists a C^d-convergent subdivision scheme with an extra regularity of p for any given $p \ge 0$. Before we give a proof by means of an explicit construction for such a scheme, we formally state the result.

Theorem 4.1. *Given $p \ge 0$ and a generalized Taylor operator T_d of order d, there exists a finitely supported mask $\boldsymbol{A} \in \ell^{r\times r}$ such that the Hermite subdivision $H_{\boldsymbol{A}}$ scheme is C^d-convergent with a limit function $\phi \in C^{d+p}(\mathbb{R})$ and $S_{\boldsymbol{A}}$ admits a Taylor factorization with respect to T_d.*

The idea behind the construction is simple and to some extent even follows the same concept as usual scalar subdivision: starting with the Taylor factor $\boldsymbol{B}$ such that $T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d$, we increase the order of smoothness of the limit function of $S_{\boldsymbol{B}}$ by constructing a scheme whose symbol has extra (matrix) factors. This means that $S_{\boldsymbol{B}}$ from the (incomplete) Taylor factorization should be further factorizable into

$$\begin{bmatrix} \boldsymbol{I}_d & \\ & \triangle \end{bmatrix}^p S_{\boldsymbol{B}} = 2^{-p} S_{\widetilde{\boldsymbol{B}}} \begin{bmatrix} \boldsymbol{I}_d & \\ & \triangle \end{bmatrix}^p, \tag{20}$$

where $\widetilde{\boldsymbol{B}}$ is also a finitely supported mask. From [13, 14] we recall the following result on smoothing limit functions.

Theorem 4.2. *The vector subdivision scheme $S_{\boldsymbol{B}}$ has C^p limit function if*

$$\begin{bmatrix} \boldsymbol{I}_d & \\ & \triangle \end{bmatrix}^{p+1} S_{\boldsymbol{B}} = \frac{1}{2^p} S_{\widetilde{\boldsymbol{B}}} \begin{bmatrix} \boldsymbol{I}_d & \\ & \triangle \end{bmatrix}^{p+1} \tag{21}$$

and $S_{\widetilde{\boldsymbol{B}}}$ is contractive. The converse does not hold.

For the construction of an appropriate $\widetilde{\boldsymbol{B}}$, we partition it as

$$\widetilde{\boldsymbol{B}}^*(z) = \begin{bmatrix} \widetilde{\boldsymbol{B}}^*_{11}(z) & \widetilde{\boldsymbol{B}}^*_{12}(z) \\ \widetilde{\boldsymbol{B}}^*_{21}(z) & \widetilde{\boldsymbol{B}}^*_{22}(z) \end{bmatrix}, \qquad \widetilde{\boldsymbol{B}}_{11} \in \ell^{d\times d}(\mathbb{R}), \widetilde{\boldsymbol{B}}_{12}, \widetilde{\boldsymbol{B}}^T_{21} \in \ell^{1\times d}(\mathbb{R}), \widetilde{\boldsymbol{B}}_{11} \in \ell^{1\times 1}(\mathbb{R})$$

Since (21) can be rewritten as

$$\begin{aligned} \boldsymbol{B}^*(z) &= \frac{1}{2^p} \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-1}-1 \end{bmatrix}^{-p-1} \begin{bmatrix} \widetilde{\boldsymbol{B}}^*_{11}(z) & \widetilde{\boldsymbol{B}}^*_{12}(z) \\ \widetilde{\boldsymbol{B}}^*_{21}(z) & \widetilde{\boldsymbol{B}}^*_{22}(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-2}-1 \end{bmatrix}^{p+1} \\ &= \frac{1}{2^p} \begin{bmatrix} \widetilde{\boldsymbol{B}}^*_{11}(z) & (z^{-2}-1)^{p+1} \widetilde{\boldsymbol{B}}^*_{12}(z) \\ (z^{-1}-1)^{-p-1} \widetilde{\boldsymbol{B}}^*_{21}(z) & (z+1)^{p+1} \widetilde{\boldsymbol{B}}^*_{22}(z) \end{bmatrix}, \end{aligned} \tag{22}$$

we can record the following immediate consequence of Theorem 4.2.

Corollary 4.3. *$S_{\boldsymbol{B}}$ converges to a C^p limit function of the form $\boldsymbol{f}_{\boldsymbol{c}} = f_{\boldsymbol{c}} \boldsymbol{e}_d$ if*

1. *$S_{\widetilde{\boldsymbol{B}}}$ is contractive,*
2. *$\widetilde{\boldsymbol{B}}^*_{21}$ has a zero of order $p+1$ at 1,*
3. *$\widetilde{\boldsymbol{B}}$ is normalized as $\widetilde{\boldsymbol{B}}^*_{22}(1) = 1$.*

The corollary tells us that contractive schemes are at the heart of the construction of a convergent Hermite subdivision scheme. Note that contractivity of a scheme $\boldsymbol{C}$ means that the spectral radius

$$\rho(S_{\boldsymbol{C}}) := \limsup_{n\to\infty} \|S^n_{\boldsymbol{C}}\|^{1/n}, \qquad \|S_{\boldsymbol{C}}\| := \sup_{\|\boldsymbol{c}\|_\infty = 1} \|S_{\boldsymbol{C}}\, \boldsymbol{c}\|_\infty,$$

based on the operator norm of the *subdivision operator* is less than one, where

$$\|\boldsymbol{c}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \max_{0\le k<r} |c_k(\alpha)|.$$

The following simple sufficient condition for contractivity of a vector subdivision scheme is most likely known in the folklore, but we state it and give a quick proof for the sake of completeness and the reader's convenience.

Lemma 4.4. *If $\boldsymbol{C}$ is a lower triangular mask, i.e., all components of $\boldsymbol{C}(\alpha)$ are lower triangular matrices and the diagonal elements $c_{00}, \dots, c_{r-1,r-1} \in \ell(\mathbb{Z})$ are scalar contractive schemes, then $\boldsymbol{C}$ defines a contractive vector subdivision scheme.*

Proof. Write $\boldsymbol{C} = \boldsymbol{D} + \boldsymbol{N}$ where $\boldsymbol{D} \in \ell^{r\times r}(\mathbb{Z})$ is a diagonal scheme and $\boldsymbol{N} \in \ell^{r\times r}(\mathbb{Z})$ is strictly lower diagonal one, then $S^n_{\boldsymbol{C}} = S^n_{\boldsymbol{D}} + S_{\boldsymbol{N}_n}$ for some strictly lower diagonal $\boldsymbol{N}_n \in \ell^{r\times r}(\mathbb{Z})$, $n \in \mathbb{N}$. Since the diagonal elements are contractions, there exist some $n \in \mathbb{N}$ such that $\left\|S^n_{\boldsymbol{D}}\right\| < 1$. With

$$\boldsymbol{E}_\varepsilon := \begin{bmatrix} 1 & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon^{r-1} \end{bmatrix}$$

we have that

$$\boldsymbol{E}_\varepsilon S_{\boldsymbol{C}}^n \boldsymbol{E}_\varepsilon^{-1} = S_{\boldsymbol{D}}^n + S_{\boldsymbol{E}_\varepsilon \boldsymbol{N}^n \boldsymbol{E}_\varepsilon^{-1}}, \qquad \boldsymbol{E}_\varepsilon \boldsymbol{N}^n \boldsymbol{E}_\varepsilon^{-1} = \begin{bmatrix} 0 & & & \\ \varepsilon * & 0 & & \\ \vdots & \ddots & \ddots & \\ \varepsilon^{r-1} * & \dots & \varepsilon * & 0 \end{bmatrix}$$

and hence there exists $\varepsilon > 0$ such that $\rho := \left\| \boldsymbol{E}_\varepsilon S_{\boldsymbol{C}}^n \boldsymbol{E}_\varepsilon^{-1} \right\| < 1$. Hence, for any $m \in \mathbb{N}$,

$$\|S_{\boldsymbol{C}}^{mn}\| \le \|\boldsymbol{E}_\varepsilon\| \|\boldsymbol{E}_\varepsilon^{-1}\| \left\| \boldsymbol{E}_\varepsilon S_{\boldsymbol{C}}^{mn} E_\varepsilon^{-1} \right\| = \varepsilon^{1-r} \left\| \left(\boldsymbol{E}_\varepsilon S_{\boldsymbol{C}}^n \boldsymbol{E}_\varepsilon^{-1} \right)^m \right\| \le \frac{\rho^r}{e^{r-1}}$$

which becomes < 1 for m sufficiently large. Since $\rho(S_{\boldsymbol{C}}) \le \|S_{\boldsymbol{C}}^{mn}\|^{1/(mn)}$ for any choice of $m, n \in \mathbb{N}$, this completes the proof that $S_{\boldsymbol{C}}$ is a contraction. □

Next, note that the second condition on $\widetilde{\boldsymbol{B}}$ in Corollary 4.3 ensures that $\boldsymbol{B}^*$ in (22) is a Laurent polynomial while the normalization yields

$$\boldsymbol{B}^*(1) = \begin{bmatrix} * & 0 \\ 0 & 2 \end{bmatrix}, \qquad \boldsymbol{B}^*(-1) = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix},$$

hence

$$\left(\sum_{\alpha \in \mathbb{Z}} \boldsymbol{B}(2\alpha) \right) \boldsymbol{e}_d = \left(\sum_{\alpha \in \mathbb{Z}} \boldsymbol{B}(2\alpha + 1) \right) \boldsymbol{e}_d = \boldsymbol{e}_d,$$

which is the necessary condition for the limit function to be of the form $f_c \boldsymbol{e}_d$. Combining the two factorizations into one, we arrive at the following definition.

Definition 4.1. The (generalized) *supercomplete Taylor operator* of order d and extra regularity p is of the form

$$T_{d,p} := \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & \Delta^{p+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_d & \\ & \Delta \end{bmatrix}^p \widetilde{T}_d = \begin{bmatrix} \boldsymbol{I}_d & \\ & \Delta \end{bmatrix}^{p+1} T_d. \tag{23}$$

The special cases are $T_d = T_{d,-1}$ and $\widetilde{T}_d = T_{d,0}$.

A factorization with respect to a supercomplete operator, $T_{d,p} S_{\boldsymbol{A}} = 2^{-p-d} S_{\widehat{\boldsymbol{B}}} T_{d,p}$ is equivalent to

$$\begin{aligned} \boldsymbol{A}^*(z) &= \frac{1}{2^{p+d}} \left(\boldsymbol{T}_{d,p}^*(z) \right)^{-1} \widehat{\boldsymbol{B}}^*(z) \widehat{\boldsymbol{T}}_{d,p}^*(z^2) \\ &= \frac{1}{2^{p+d}} \left(\widetilde{\boldsymbol{T}}^*(z) \right)^{-1} \begin{bmatrix} \boldsymbol{I}_d & \\ & (z^{-1} - 1)^{-p} \end{bmatrix} \widehat{\boldsymbol{B}}^*(z) \begin{bmatrix} \boldsymbol{I}_d & \\ & (z^{-2} - 1)^p \end{bmatrix} \widetilde{\boldsymbol{T}}^*(z^2) \\ &= \frac{1}{2^{p+d}} \left(\widetilde{\boldsymbol{T}}^*(z) \right)^{-1} \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-1} - 1 \end{bmatrix}^{-p-1} \widehat{\boldsymbol{B}}^*(z) \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-2} - 1 \end{bmatrix}^{p+1} \widetilde{\boldsymbol{T}}^*(z^2) \\ &=: \frac{1}{2^d} \left(\widetilde{\boldsymbol{T}}^*(z) \right)^{-1} \boldsymbol{C}^*(z) \widetilde{\boldsymbol{T}}^*(z^2). \end{aligned}$$

Thus, if we can find mask $\boldsymbol{C}$ associated to a contractive scheme and normalized as $(\boldsymbol{C}^*(1))_{dd} = 1$, such that $H_{\boldsymbol{A}}$ is a C^d-convergent subdivision scheme, then we can compute $\widehat{\boldsymbol{B}}^*(z)$ as

$$\begin{aligned} \widehat{\boldsymbol{B}}^*(z) &= 2^p \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-1} - 1 \end{bmatrix}^{p+1} \boldsymbol{C}^*(z) \begin{bmatrix} \boldsymbol{I}_d & \\ & z^{-1} - 1 \end{bmatrix}^{-p-1} \\ &= 2^p \begin{bmatrix} \widetilde{\boldsymbol{C}}_{11}^*(z) & \left(\frac{1}{z^{-2} - 1} \right)^{p+1} \widetilde{\boldsymbol{C}}_{12}^*(z) \\ (z^{-1} - 1)^{p+1} \widetilde{\boldsymbol{C}}_{21}^*(z) & \left(\frac{1}{z^{-1} + 1} \right)^{p+1} \widetilde{\boldsymbol{C}}_{22}^*(z) \end{bmatrix}. \end{aligned}$$

The construction of $\boldsymbol{C}$ has been pointed out in [11]. More precisely, given any symbol h_{dd}^* of a mask h such that the univariate scalar stationary subdivision scheme S_h is contractive, then there exists a recursive scheme [11, eq. (65) in the proof of Theorem 5] to compute $h_{d,d-1}^*, \dots, h_{d,0}$ such that for any h_{jk}^*, $k = 0, \dots, j-1$, $j = 1, \dots, d-1$, the lower triangular symbol

$$\boldsymbol{C}^*(z) = \begin{bmatrix} \frac{z^{-1}-1}{2} & & & & \\ (z^{-1} - 1)^2 h_{10}^*(z) & \frac{(z^{-1}-1)^2}{4} & & & \\ \vdots & \ddots & \ddots & & \\ (z^{-1} - 1)^d h_{d-1,0}^*(z) & \dots & (z^{-1} - 1)^d h_{d-1,d-2}^*(z) & \frac{(z^{-1}-1)^d}{2^d} & \\ c_{d0}^*(z) & \dots & c_{d,d-2}^*(z) & c_{d,d-1}^*(z) & c_{dd}^*(z) \end{bmatrix}$$

with

$$c_{dj}^{*}(z) = \left(z^{-1} - 1\right)^{d-j} h_{dj}^{*}(z^{-1}), \qquad j = 0, \dots, d,$$

defines a Taylor factor with a contractive associated subdivision scheme by Lemma 4.4. Note, in particular, that $\boldsymbol{C}_{12}^{*} = 0$ and that $c_{dd}^{*} = h_{dd}^{*}$. If, in addition, we choose

$$h_{dd}^{*}(z) = \frac{(z+1)^{p+1}}{2^{p+1}} a(z), \qquad a(1) = 2,$$

in a B-spline fashion, then $\widehat{\boldsymbol{B}}^{*}(z)$ is a matrix Laurent polynomial and $S_{\boldsymbol{A}}$, defined by

$$\boldsymbol{A}^{*}(z) \quad = \quad \frac{1}{2^{d}} \left(\widetilde{\boldsymbol{T}}^{*}(z)\right)^{-1} \boldsymbol{C}^{*}(z) \widetilde{\boldsymbol{T}}^{*}(z^{2}) = \frac{1}{2^{p+d}} \left(\boldsymbol{T}_{d,p}^{*}(z)\right)^{-1} \widehat{\boldsymbol{B}}^{*}(z) \widehat{\boldsymbol{T}}_{d,p}^{*}(z^{2})$$

defines a C^{d} convergent Hermite subdivision scheme by Theorem 2.1 and has a p-supercomplete Taylor factorization with factor $\widehat{\boldsymbol{B}}$. The symbol $\widehat{\boldsymbol{B}}^{*}$ is a lower triangular matrix with the same diagonal structure as $\boldsymbol{C}^{*}$ and thus also defines a contraction. Hence, by Theorem 4.2 and Corollary 4.3, the last component of the limit $\left[\phi, \phi', \dots, \phi^{(d)}\right]$ of $H_{\boldsymbol{A}}$ belongs to $C^{p}(\mathbb{R})$ which eventually verifies that $\phi \in C^{d+p}(\mathbb{R})$.

This also concludes the proof of Theorem 4.2.

We finish with revisiting one example from [11] where already a mask with a supercomplete Taylor factorization was constructed.

Example 4.1. In the case $n = 5$ with the functions from [11, Example 5], we can get a factorization with $p = 4$ and obtain

$$\widehat{\boldsymbol{B}}^{*}(z) \quad = \quad \begin{bmatrix} -\frac{8(z-1)}{z} & 0 & 0 \\ \frac{16(z-1)^{2}}{z^{2}} & \frac{4(z-1)^{2}}{z^{2}} & 0 \\ \frac{32(z-1)^{6}(5-4a-3z+4az)}{z^{7}} & -\frac{8(z-1)^{5}\left(15-20a+16a^{2}-18z+32az-32a^{2}z+7z^{2}-12az^{2}+16a^{2}z^{2}\right)}{z^{7}} & \frac{1+z}{2z} \end{bmatrix}$$

as well as

$$\boldsymbol{A}^{*}(z)$$

$$= \begin{bmatrix} -\frac{(1+z)\left(-20a+16a^{2}-23z+48az-32a^{2}z+15z^{2}-28az^{2}+16a^{2}z^{2}\right)}{2z^{3}} & \frac{30a-40a^{2}+32a^{3}+31z-46az+56a^{2}z-32a^{3}z+24z^{2}-78az^{2}+72a^{2}z^{2}-32a^{3}z^{2}-45z^{3}+94az^{3}-88a^{2}z^{3}+32a^{3}z^{3}}{4z^{3}} & \frac{a+z+7az+8z^{2}+22az^{2}+30z^{3}+42az^{3}+72z^{4}-183az^{4}-119z^{5}+63az^{5}}{32z^{5}} \\ -\frac{(z-1)(1+z)(11-8a-7z+8az)}{z^{3}} & \frac{(z-1)\left(-31+40a-32a^{2}-24z+16az+43z^{2}-56az^{2}+32a^{2}z^{2}\right)}{4z^{3}} & \frac{(z-1)\left(-1-8z-30z^{2}-72z^{3}+119z^{4}+32az^{4}\right)}{32z^{5}} \\ \frac{2(z-1)^{2}(1+z)(5-4a-3z+4az)}{z^{4}} & -\frac{(z-1)^{2}\left(-15+20a-16a^{2}-12z+8az+19z^{2}-28az^{2}+16a^{2}z^{2}\right)}{2z^{4}} & -\frac{(z-1)^{3}\left(1+9z+39z^{2}+111z^{3}\right)}{32z^{6}} \end{bmatrix}.$$

In this expression, a is the free parameter of the associated generalized Taylor operator with complete form

$$\widetilde{T}_{d} = \begin{bmatrix} \Delta & -1 & a \\ & \Delta & -1 \\ & & \Delta \end{bmatrix}.$$

In Fig. 2, we have plotted the "limit" function and its first and second derivatives. Since the process does not compute the next derivatives, the approximations for higher derivatives in Fig. 3 have been determined using the successive finite differences of $f^{(2)}$.

5 Conclusion

Convergent Hermite subdivision schemes of order d have a limit function that belongs to C^{d}, at least if the scheme converges in the sense proper for Hermite subdivision. We have shown that this regularity can be raised to an arbitrary order and provided an explicit recipe to determine such schemes. The approach uses factorizations and contractivity, but in contrast to the scalar univariate case the factor mask $\widetilde{\boldsymbol{B}}$ must satisfy additional, nontrivial conditions, and the main task in the construction is to satisfy these conditions.

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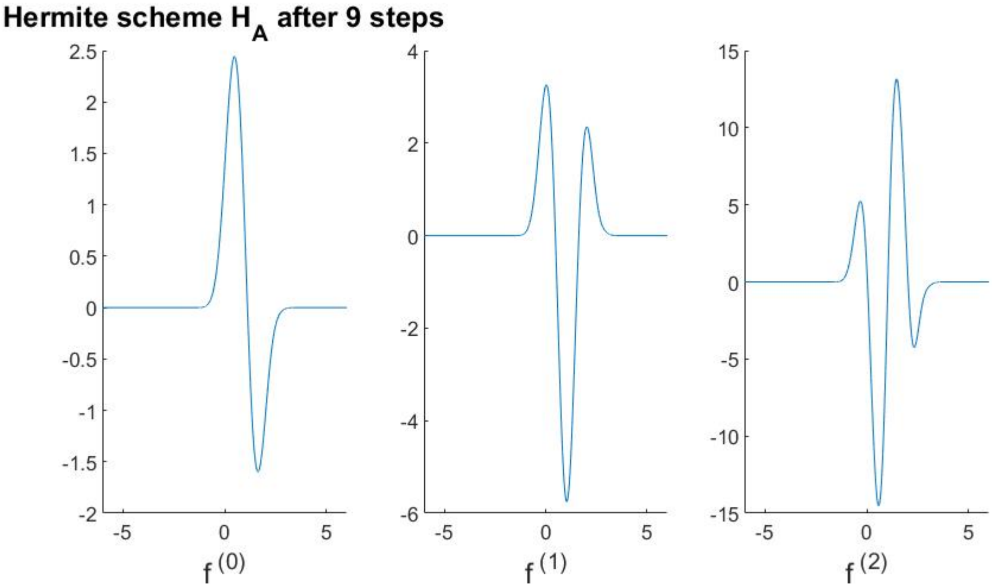

Figure 2: The function and its first and second derivatives

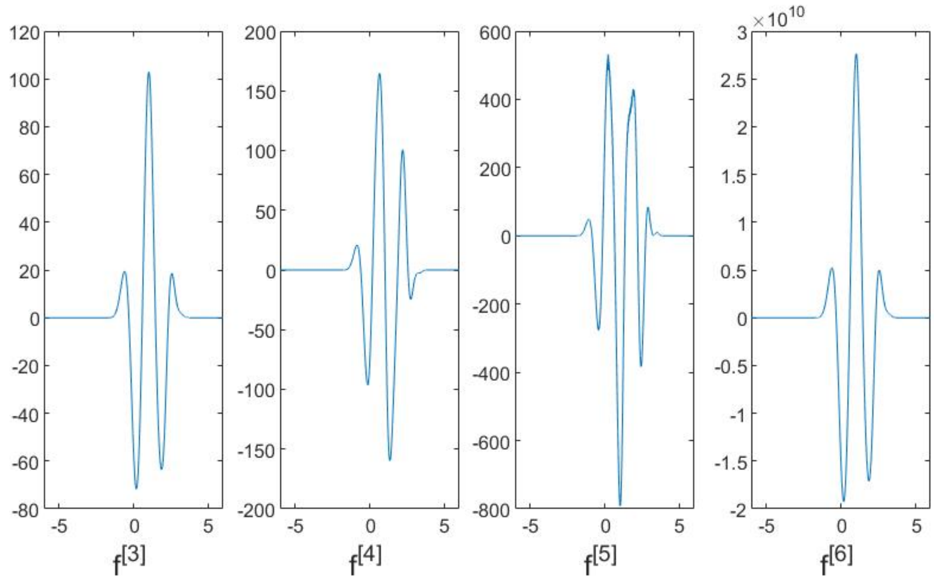

Figure 3: Approximation of the successive derivatives by finite differences

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