The length of the de Rham curve

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Abstract

The length L of the de Rham curve is the common limit of two monotonic sequences of lengths (l^n) and (L^n) of inscribed and circumscribed polygons respectively. Numerical computations show that their convergence is linear with the same convergence rate. This result is easy to prove for the parabola. For arbitrary de Rham curves, we prove two nearby results. Firstly, the existence of a limit $q \in]0,1[$ of the sequence of ratios $(L^{n+1} - L)/(L^n - L)$ implies the convergence to the same limit of the two sequences $(l^{n+1} - L)/(l^n - L)$ and $(L^{n+1} - l^{n+1})/(L^n - l^n)$. Secondly, the sequence $(L^{n+1} - L^n)$ is bounded by a convergent geometric sequence. In practice, this allows to accelerate the convergence of both sequences by standard extrapolation algorithms.

1. Introduction

The de Rham curve C_γ, studied in [3], is the limit of a sequence of polygons depending on a parameter γ.
We are interested in the computation of the length L of this curve. This problem was already considered by other authors in the context of computer-aided geometric design, in particular for piecewise polynomial or rational curves (see e.g. [4],[5]). In a further paper, we shall develop applications to the problem of constructing an interpolating convex curve with prescribed length.

Here is an outline of the paper: in Section 2, we recall the construction of the curve C_γ and its known properties. In Section 3, we first define upper

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and lower approximations of L as the lengths of two approximating polygons. They form two sequences (L^n) and (l^n) which both converge monotonically to L. In Section 4, we study the convergence speed of (L^n) to L. Numerical computations strongly suggest that both sequences (L^n) and (l^n) converge lineary to L for all $\gamma > 1$ (various examples are given in Section 6). We prove that the existence of $q = \lim_{n\to+\infty} \frac{L^{n+1} - L^n}{L^n - L^{n-1}}$, $q \in]0, 1[$, with $q \neq \frac{\gamma}{\gamma+2}$, implies the existence of $\lim_{n\to+\infty} \frac{L^{n+1} - L}{L^n - L}$ and $\lim_{n\to+\infty} \frac{l^{n+1} - L}{l^n - L}$; moreover these limits are also q. But we did not succeed in proving the existence of this limit, except for the parabola ($\gamma = 2$). In that case, we prove in Section 5 that both limits are equal to $1/4$. However, in the general case, ($\gamma > 1$), we can prove that there exist constants $c > 0$ and $0 < \kappa < 1$ depending on γ such that $|L^{n+1} - L^n| \leq c\kappa^n$. This shows that the convergence of (L^n) is at worst linear.

Finally, this suggests the possibility of accelerating the convergence of the two sequences (L^n) and (l^n) by the ε-algorithm or the iterated Aitken's Δ^2 algorithm (see e.g. chapter 2 of [1]), since they do not need the knowledge of the exact rate of convergence of these sequences.

2. Construction and properties of the de Rham curve

Let ABC be a triangle. The curve is the limit of a sequence of polygons, $\{P^n, n = 0, 1, 2, \ldots\}$ starting with $P^0 = \{A, B, C\}$. Then the points dividing in three parts the sides of the polygon P^n obtained at the n-th step are the vertices of the next one. The three parts have lengths proportional to $1, \gamma, 1$ successively. The number of sides of P^n is $2^n + 1$.

We denote by $S_0^n, S_1^n, \ldots, S_{2^n+1}^n$ the vertices of P^n. The construction of de Rham in order to get the next polygon $P^{n+1} = \{S_0^{n+1}, S_1^{n+1}, \ldots, S_{2^{n+1}+1}^{n+1}\}$ from the previous one P^n is as follows: $S_{2i}^{n+1} = (1-\beta)S_i^n + \beta S_{i+1}^n$ and $S_{2i+1}^{n+1} = \beta S_i^n + (1-\beta)S_{i+1}^n$ for $i = 0 \ldots 2^n$, where $\beta = 1/(\gamma + 2)$.

In Fig. 1, we show the first step in the construction of de Rham with $P^0 = \{A, B, C\}$ and $P^1 = \{A', B', C', D'\}$.

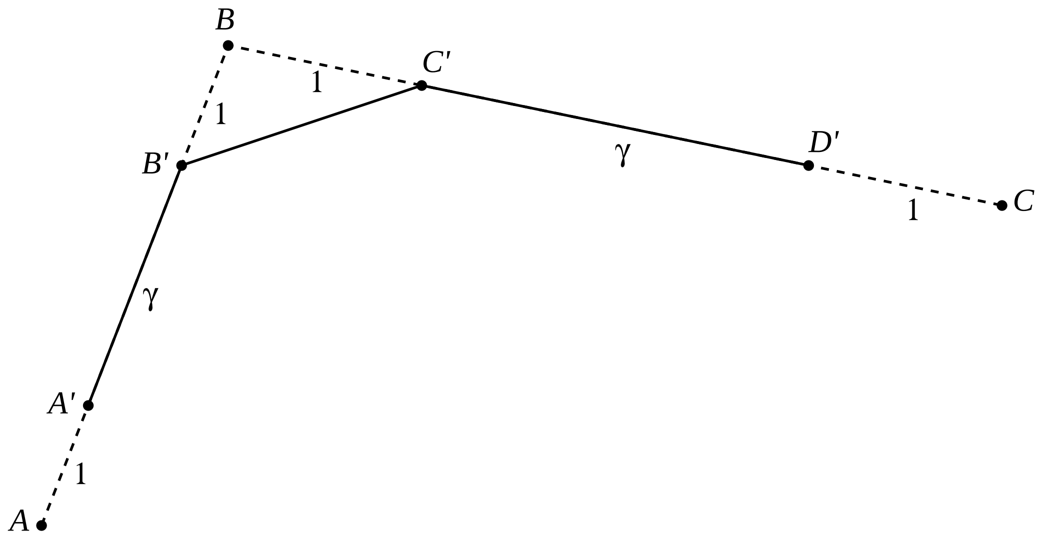

Figure 1: The first step in the construction of de Rham

The following properties are given by de Rham.

- The polygons P^n are convex and the sequence (P^n) converges to a curve C_γ which is continuous and convex.
- C_γ is tangent at the midpoint of each side of P^n.
- If $\gamma > 1$, C_γ has a tangent at each point and the slope m is continuous.
- For $\gamma = 2$, C_2 is an arc of a parabola from the midpoint of $[AB]$ to the midpoint of $[BC]$.

For the next sections, we shall suppose $\gamma > 1$.

3. Upper and lower approximations of the length of the curve

We denote by $M_0^n, M_1^n, \ldots, M_{2^n}^n$ the midpoints of the sides of P^n. Let L^n be the length of P^n measured from the midpoint M_0^n of the first side to the midpoint $M_{2^n}^n$ of the last one, and let l^n be the length of the polygonal line joining the midpoints: $M_0^n M_1^n \ldots M_{2^n}^n$.With these notations, $M_i^n = (S_i^n + S_{i+1}^n)/2$ and $M_{2i}^{n+1} = M_i^n$. We write $|U|$ for the euclidean norm of the vector U. Thus, we have

$L^0 = |M_0^0 S_1^0| + |S_1^0 M_1^0| = (|AB| + |BC|)/2,$

$l^0 = |AC|/2,$

and for all $n \in \mathbb{N}$

$L^n = \sum_{i=0}^{2^n-1} |M_i^n S_{i+1}^n| + |S_{i+1}^n M_{i+1}^n|$

$l^n = \sum_{i=0}^{2^n-1} |M_i^n M_{i+1}^n|$

From now on, we shall omit the upper index n in M_i^n if it is not necessary for the comprehension; similarly, we shall write M'_j instead of M_j^{n+1}.

Proposition 1 *For all $n \in \mathbb{N}$, there holds*

$$L^{n+1} = \frac{\gamma L^n + 2l^n}{\gamma + 2}$$

Proof : Let $j = 2i$, then for $i = 0 \ldots 2^n$ we have,

$|M'_j S'_{j+1}| + |S'_{j+1} M'_{j+1}| + |M'_{j+1} S'_{j+2}| + |S'_{j+2} M'_{j+2}|$

$= |M'_j S'_{j+1}| + |S'_{j+1} S'_{j+2}| + |S'_{j+2} M'_{j+2}|$

$= \dfrac{\gamma}{\gamma+2}|M_i S_{i+1}| + \dfrac{2}{\gamma+2}|M_i M_{i+1}| + \dfrac{\gamma}{\gamma+2}|S_{i+1} M_{i+1}|$

$= \dfrac{\gamma}{\gamma+2}(|M_i S_{i+1}| + |S_{i+1} M_{i+1}|) + \dfrac{2}{\gamma+1}|M_i M_{i+1}|$

Adding these equalities gives the result, since

$L^{n+1} = \sum_{j=0}^{2^{n+1}-1} |M'_j S'_{j+1}| + |S'_{j+1} M'_{j+1}|.$ □

Proposition 2 *The two sequences (L^n) and (l^n) are respectively decreasing and increasing and they converge to the same limit L, which is the length of C_γ.*

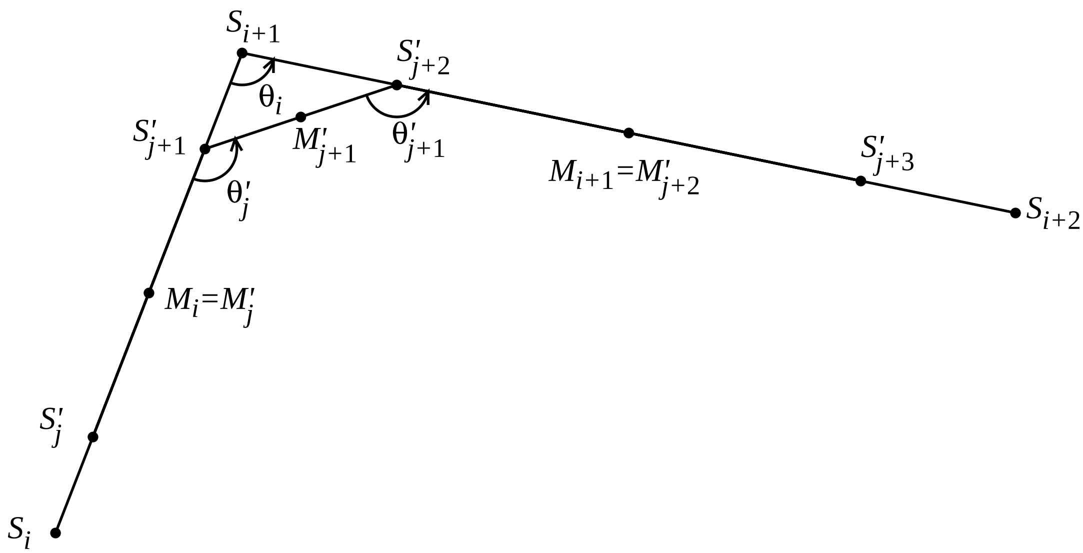

Figure 2: The main parameters in the construction of de Rham

Proof : Let again $j = 2i$

- $|M_i M_{i+1}| = |M'_j M'_{j+2}| \leq |M'_j M'_{j+1}| + |M'_{j+1} M'_{j+2}|$,
  therefore $l^n \leq l^{n+1}$.

- similary
  $|M_i S_{i+1}| + |S_{i+1} M_{i+1}|$
  $= |M_i S'_{j+1}| + |S'_{j+1} S_{i+1}| + |S_{i+1} S'_{j+2}| + |S'_{j+2} M_{i+1}|$
  $\geq |M_i S'_{j+1}| + |S'_{j+1} S'_{j+2}| + |S'_{j+2} M_{i+1}|$
  $= |M'_j S'_{j+1}| + |S'_{j+1} M'_{j+1}| + |M'_{j+1} S'_{j+2}| + |S'_{j+2} M'_{j+2}|$,
  therefore $L^n \geq L^{n+1}$.

- Now $L^n \geq |AC|/2$ and $l^n \leq (|AB| + |BC|)/2$, so that both sequences are converging respectively to L and l. From the preceding proposition, we deduce $L = l$, which is the length of C_γ.

□

4. On the rate of convergence of the sequence L^n

We shall now study the asymptotic behavior of the sequence (L^n); we first prove that when n goes to $+\infty$ the ratios of the lengths of any two consecutive sides of P^n are uniformly bounded, then we prove that the angle between two sides tends to π. With the help of these preliminary results we shall be able to study the convergence of (L^n).

We denote by $\lambda_0 = |S_0S_1|, \lambda_1 = |S_1S_2|, \dots, \lambda_{2^n} = |S_{2^n}S_{2^n+1}|$ the successive lengths of the sides of P^n and by $\lambda'_0, \lambda'_1, \dots, \lambda'_{2^{n+1}}$ those of P^{n+1}. The sides of P^n (resp P^{n+1}) make angles $\theta_0 = \angle(S_0S_1, S_1S_2), \dots, \theta_{2^n-1} = \angle(S_{2^n-1}S_{2^n}, S_{2^n}S_{2^n+1})$ (resp $\theta'_0, \dots, \theta'_{2^{n+1}-1}$). See fig 2.

We shall study the ratio $\dfrac{L^{n+1} - L^n}{L^n - L^{n-1}}$; indeed, Brezinski and Redivo Zaglia have proved in [1] that for a sequence (u^n) converging to $u \in \mathbb{C}$ and for $q \in \mathbb{C}$ with $|q| \neq 1$, $\lim\limits_{n\to+\infty} \dfrac{u^{n+1} - u}{u^n - u} = q$ if and only if $\lim\limits_{n\to+\infty} \dfrac{u^{n+1} - u^n}{u^n - u^{n-1}} = q$.

With the next proposition, we shall be able to get the rate of convergence of (l^n) from the one of (L^n).

Proposition 3 *If* $\lim\limits_{n\to+\infty} \dfrac{L^{n+1} - L}{L^n - L} = q$ *with* $q \neq 1$ *and* $q \neq \dfrac{\gamma}{\gamma+2}$ *then* $\lim\limits_{n\to+\infty} \dfrac{l^{n+1} - L}{l^n - L} = \lim\limits_{n\to+\infty} \dfrac{L^{n+1} - l^{n+1}}{L^n - l^n} = q$

Proof : By Proposition 1, $(\gamma + 2)L^{n+1} = \gamma L^n + 2l^n$,
then $(\gamma + 2)L = \gamma L + 2L$. So that, by difference
$(\gamma + 2)(L^{n+1} - L) - \gamma(L^n - L) = 2(l^n - L)$ and
$(\gamma + 2)(L^n - L) - \gamma(L^{n-1} - L) = 2(l^{n-1} - L)$ from the preceding step.
By division, we successively get:

$$\frac{(\gamma + 2)(L^{n+1} - L) - \gamma(L^n - L)}{(\gamma + 2)(L^n - L) - \gamma(L^{n-1} - L)} = \frac{l^n - L}{l^{n-1} - L}$$

$$\frac{(\gamma+2)(L^{n+1}-L)/(L^n-L)-\gamma}{\gamma+2-\gamma(L^{n-1}-L)/(L^n-L)} = \frac{l^n-L}{l^{n-1}-L}.$$

As n tends to $+\infty$, we get

$$q = \frac{(\gamma+2)q-\gamma}{\gamma+2-\gamma/q} = \lim_{n\to+\infty} \frac{l^{n+1}-L}{l^n-L}$$

as soon as $q \neq \dfrac{\gamma}{\gamma+2}$.
Similarly, using $(\gamma+2)[(L^{n+1}-L)-(L^n-L)] = 2(l^n-L^n)$ and the preceding step, we easily get

$$q = \frac{q-1}{1-1/q} = \lim_{n\to+\infty} \frac{l^n-L^n}{l^{n-1}-L^{n-1}}$$

as soon as $q \neq 1$. □

Proposition 4 *For every $n \in \mathbb{N}$ we have*

$$L^{n+1}-L^n = \frac{1}{\gamma+2}\sum_{i=0}^{2^n-1}(\lambda_i+\lambda_{i+1})\Big(\sqrt{1-\frac{4\lambda_i\lambda_{i+1}\cos^2(\theta_i/2)}{(\lambda_i+\lambda_{i+1})^2}}-1\Big).$$

Proof :

We recall that $\beta = 1/(\gamma+2)$, we set $\alpha = \gamma/(\gamma+2)$ and again $j = 2i$. Then for $i = 0, 1, \ldots, 2^n-1$, writing $a_i = \beta\lambda_i$ and $b_i = \beta\lambda_{i+1}$, we have

$$\begin{aligned}
\lambda'_j &= \alpha\lambda_i \\
\lambda'_{j+1} &= \sqrt{a_i^2+b_i^2-2a_ib_i\cos\theta_i} \\
&= \sqrt{(a_i+b_i)^2-4a_ib_i\cos^2\frac{\theta_i}{2}} \\
&= \beta\sqrt{(\lambda_i+\lambda_{i+1})^2-4\lambda_i\lambda_{i+1}\cos^2\frac{\theta_i}{2}} \\
&= \beta(\lambda_i+\lambda_{i+1})\sqrt{1-\frac{4\lambda_i\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^2}\cos^2\frac{\theta_i}{2}}
\end{aligned}$$

and $\lambda'_{2^{n+1}} = \alpha\lambda_{2^n}$

Using $L^n = \lambda_0/2 + \sum_{i=1}^{2^n-1}\lambda_i + \lambda_{2^n}/2$ and a similar formula for L^{n+1}, we can evaluate $\varepsilon^n = L^{n+1}-L^n$.

$$
\begin{aligned}
\varepsilon^n &= \alpha\lambda_0/2 + \alpha\lambda_1 + \ldots + \alpha\lambda_{2^n-1} + \alpha\lambda_{2^n}/2 \\
&+ \sum_{i=0}^{2^n-1} \beta(\lambda_i + \lambda_{i+1})\sqrt{1 - \frac{4\lambda_i\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^2}\cos^2\frac{\theta_i}{2}} \\
&- (\lambda_0/2 + \lambda_1 + \ldots + \lambda_{2^n-1} + \lambda_{2^n}/2) \\
&= \sum_{i=0}^{2^n-1} \Big(\beta(\lambda_i + \lambda_{i+1})\sqrt{1 - \frac{4\lambda_i\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^2}\cos^2\frac{\theta_i}{2}} + \frac{(\alpha-1)(\lambda_i+\lambda_{i+1})}{2}\Big) \\
&= \beta\sum_{i=0}^{2^n-1}(\lambda_i + \lambda_{i+1})\Big(\sqrt{1 - \frac{4\lambda_i\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^2}\cos^2\frac{\theta_i}{2}} - 1\Big)
\end{aligned}
$$

□

Proposition 5 *Let* $r_i^n = \dfrac{\lambda_{i+1}^n}{\lambda_i^n}$ *for* $i \in \{0, \ldots, 2^n\}$ *be the ratios of two successive lengths at step* n*. If* $\gamma > 1$*, then there exist* r^0 *and* R^0 *such that: for every* $n \in \mathbb{N}$*, for every* $i \in \{0, \ldots, 2^n\}, r^0 \leq r_i^n \leq R^0$.

Proof : First, let us remark that the angles θ_i^n of P^n are bounded away from 0 since $\theta_0^0 \leq \theta_i^n \leq \pi$. We can suppose that there exists $\theta^0 \in (0, \pi)$ such that:

$$\theta^0 \leq \theta_i^n \leq \pi$$

Now, omitting n and i, let $r = \dfrac{\lambda_{i+1}}{\lambda_i}$, $\theta = \angle(S_iS_{i+1}, S_{i+1}S_{i+2})$ and
$r'_1 = \dfrac{\lambda'_{j+1}}{\lambda'_j}$, $r'_2 = \dfrac{\lambda'_{j+2}}{\lambda'_{j+1}}$, with $j = 2i$.
Then $r'_1 = \dfrac{1}{\gamma}(r^2 - 2r\cos\theta + 1)^{\frac{1}{2}}$ and $r'_2 = \dfrac{\gamma r}{(r^2 - 2r\cos\theta + 1)^{\frac{1}{2}}}$, see fig 2.
We consider the two functions f and g defined by:
$f(x) = \dfrac{1}{\gamma}(x^2 - 2x\cos\theta + 1)^{\frac{1}{2}}$ and $g(x) = \gamma x(x^2 - 2x\cos\theta + 1)^{-\frac{1}{2}}$ with $x > 0$.
So we have $r'_1 = f(r)$ and $r'_2 = g(r)$. Since

$$
\begin{aligned}
x^2 - 2x\cos\theta + 1 &\leq x^2 + 2x + 1 = (1+x)^2 \\
x^2 - 2x\cos\theta + 1 &\geq x^2 - 2x\cos\theta^0 + 1 \geq \sin^2\theta^0,
\end{aligned}
$$

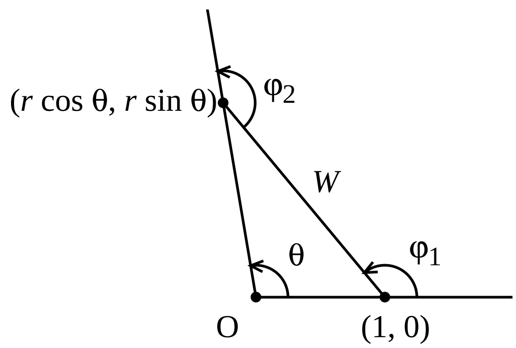

Figure 3: Computation of angles in the construction of de Rham

we get the following bounds for $f(x), x \in \mathbb{R}^+$:

$$\frac{\sin\theta^0}{\gamma} \leq f(x) \leq \frac{1+x}{\gamma} \leq \max(x, \frac{1}{\gamma - 1}),$$

and similarly, we obtain:

$$\min(x, \gamma - 1) \leq g(x) \leq \frac{\gamma}{\sin\theta^0}.$$

Now let $r^0 = \min\left(\frac{|BC|}{|AB|}, \frac{\sin\theta^0}{\gamma}, \gamma - 1\right)$ and $R^0 = \max\left(\frac{|BC|}{|AB|}, \frac{\gamma}{\sin\theta^0}, \frac{1}{\gamma-1}\right)$. Using the above inequalities,we immediately get the result by induction on $n \in \mathbb{N}$. $\square$

Proposition 6 *There exists $c \in \mathbb{R}$ and $q \in]0,1[$ such that for every $n \in \mathbb{N}$, for every $i \in \{1, \ldots, 2^n\}, |\pi - \theta_i^n| \leq cq^n$*

Proof :

In the following computations, we are using Fig. 2 and Fig 3. Let us consider in the polygonal line P^n the triangle $S_i S_{i+1} S_{i+2}$. We set $\theta = \angle(S_i S_{i+1}, S_{i+1} S_{i+2})$. If $r = |S_{i+1} S_{i+2}|/|S_i S_{i+1}|$, then the triangle $S_i S_{i+1} S_{i+2}$ is similar to the triangle whose vertices are $(1,0)$, $(0,0)$, $(r\cos\theta, r\sin\theta)$. If $j = 2i$, we denote by $\varphi_1 = \angle(S'_j S'_{j+1}, S'_{j+1} S'_{j+2})$ and $\varphi_2 = \angle(S'_{j+1} S'_{j+2}, S'_{j+2} S'_{j+3})$.

Since the triangle $S'_j S'_{j+1} S'_{j+2}$ is similar to $S_i S_{i+1} S_{i+2}$, the vector W whose endpoints are $(1,0)$ and $(r\cos\theta, r\sin\theta)$ makes an angle φ_1 with the x-axis. If $w = |W|$, one has the vector identity $(w\cos\varphi_1, q\sin\varphi_1) = (r\cos\theta - 1, r\sin\theta)$. From this, it follows that

$$\cot\varphi_1 = \cot\theta - \frac{1}{r\sin\theta} = \cot\theta - \frac{1}{r}\sqrt{1+\cot^2\theta}.$$

Moreover $\varphi_2 = \theta - (\pi - \varphi_1)$, hence

$$\cot\varphi_2 = \frac{1+\cot\theta\cot\varphi_1}{\cot\varphi_1 - \cot\theta} = \cot\theta - r\sqrt{1+\cot^2\theta}.$$

Setting $m_i = \cot\theta_i$, $m'_j = \cot\theta'_j$, $m'_{j+1} = \cot\theta'_{j+1}$ and $r_i = \lambda_{i+1}/\lambda_i$, with $j = 2i$, we get
$m'_j = m_i - \dfrac{\sqrt{1+{m_i}^2}}{r_i}$ and $m'_{j+1} = m_i - r_i\sqrt{1+{m_i}^2}$.
Also, $\sqrt{1+m_i^2} \geq 1$ and $\sqrt{1+m_i^2} \geq -m_i$ imply:
$m'_j \leq m_i - 1/r_i$, $m'_{j+1} \leq m_i - r_i$,
$m'_j \leq m'_i(1+1/r_i)$ and $m'_{j+1} \leq m_i(1+r_i)$.
Let $\rho = \min(1/R^0, r^0)$ where R^0 and r^0 are defined in the preceding proposition. By induction: there exists $c_1 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $i \in \{1,\ldots,2^n\}$, $m_i^n \leq c_1 - n\rho$, so that (m_i^n) tends to $-\infty$ as n tends to $+\infty$ and the angles θ_i^n tend to π.
Similarly, there exists $c_2 < 0$ such that for all n sufficiently large and for $i \in \{1,\ldots,2^n\}$, $m_i^n \leq c_2(1+\rho)^n$.
Then there exists a constant $c_3 > 0$ such that
$\sin(\pi - \theta_i^n) = \dfrac{1}{\sqrt{1+(m_i^n)^2}} \leq c_3\dfrac{1}{(1+\rho)^n} = c_3 q^n$ with $q \in]0,1[$ and we can conclude that for some constant $c_4 > 0$ there holds

$$|\pi - \theta_i^n| \leq c_4 q^n.$$

□

Proposition 7 *There exists $c \in \mathbb{R}_+^*$ and $\kappa \in]0,1[$ such that for every $n \in \mathbb{N}$,*

$$|L^{n+1} - L^n| \leq c\kappa^n$$

Proof : From Proposition 4 we know that

$$\varepsilon^n = L^{n+1} - L^n = \frac{1}{\gamma+2}\sum_{i=0}^{2^n-1}(\lambda_i+\lambda_{i+1})\Big(\sqrt{1-\frac{4\lambda_i\lambda_{i+1}\cos^2(\theta_i/2)}{(\lambda_i+\lambda_{i+1})^2}}-1\Big)$$

Now as $\sqrt{1-x} \geq 1-x$ for $x \in [0,1]$, we get

$$\begin{aligned}
|\varepsilon^n| &\leq \frac{1}{\gamma+2}\sum_{i=0}^{2^n-1}\frac{4\lambda_i\lambda_{i+1}\cos^2(\theta_i/2)}{\lambda_i+\lambda_{i+1}} \\
&\leq \frac{2}{\gamma+2}\sum_{i=0}^{2^n-1}(\lambda_i+\lambda_{i+1})\sin^2\frac{\pi-\theta_i}{2} \\
&\leq \frac{2}{\gamma+2}2L\frac{c_4 q^{2n}}{2} = c\kappa^n
\end{aligned}$$

□

5. The case of the parabola, $\gamma = 2$

Starting with the triangle ABC, for $\gamma = 2$, we obtain an arc of parabola C_2 joining the midpoint S_0 of $[AB]$ to the midpoint S_2 of $[BC]$. Let $S_1 = B$, $\Delta S_i = S_{i+1} - S_i$ and $\Delta^2 S_i = \Delta S_{i+1} - \Delta S_i$. In this particular case, we are able to evaluate the length L of C_2 and to estimate the convergence rates of the sequences (L^n) and (l^n).

Proposition 8 *The length L of the curve C_2 is equal to:*

$$\begin{aligned}
L &= \frac{|\Delta S_0|^2|\Delta S_1|^2 - (\Delta S_0.\Delta S_1)^2}{|\Delta^2 S_0|^3}\ln\Big(\frac{\Delta S_1.\Delta^2 S_0 + |\Delta S_1||\Delta^2 S_0|}{\Delta S_0.\Delta^2 S_0 + |\Delta S_0||\Delta^2 S_0|}\Big) \\
&+ \frac{|\Delta S_1|(\Delta S_1.\Delta^2 S_0) - |\Delta S_0|(\Delta S_0.\Delta^2 S_0)}{|\Delta^2 S_0|^2}.
\end{aligned}$$

Proof : The equation of the parabola is
$M(t) = S_0(1-t)^2 + 2S_1 t(1-t) + S_2 t^2$ with $t \in [0,1]$.

Since $M'(t) = 2(\Delta S_0(1-t) + \Delta S_1 t)$, we obtain:
$L = \int_0^1 |M'(t)| dt = 2\int_0^1 (p(t))^{\frac{1}{2}} dt$
where $p(t) = \alpha_0(1-t)^2 + 2\alpha_1 t(1-t) + \alpha_2 t^2$,
$\alpha_0 = |\Delta S_0|^2$, $\alpha_1 = \Delta S_0.\Delta S_1$, and $\alpha_2 = |\Delta S_1|^2$.
Let $\Delta\alpha_0 = \alpha_1 - \alpha_0 = \Delta S_0.\Delta^2 S_0$, $\Delta\alpha_1 = \alpha_2 - \alpha_1 = \Delta S_1.\Delta^2 S_0$,
$\Delta^2\alpha_0 = \alpha_2 - 2\alpha_1 + \alpha_0 = |\Delta^2 S_0|^2$, then we have
$\alpha_0\alpha_2 - \alpha_1^2 = |\Delta S_0|^2 |\Delta S_1|^2 - (\Delta S_0.\Delta S_1)^2 \geq 0$ by Schwarz inequality.

Now, let $a^2 = \dfrac{\alpha_0\alpha_2 - \alpha_1^2}{\Delta^2\alpha_0}$, then the change of variable
$u = t\sqrt{\Delta^2\alpha_0} + \dfrac{\Delta\alpha_0}{\sqrt{\Delta^2\alpha_0}}$ in the integral gives:
$L = \dfrac{2}{\sqrt{\Delta^2\alpha_0}} \int_{u_0}^{u_1} \sqrt{u^2 + a^2}\, du$,
with $u_0 = \dfrac{\Delta\alpha_0}{\sqrt{\Delta^2\alpha_0}}$ and $u_1 = \dfrac{\Delta\alpha_1}{\sqrt{\Delta^2\alpha_0}}$, from which we deduce immediately:

$$L = \frac{a^2}{\sqrt{\Delta^2\alpha_0}}\left(\ln\left(\frac{u_1 + \sqrt{u_1^2 + a^2}}{u_0 + \sqrt{u_0^2 + a^2}}\right) + \frac{u_1}{a^2}\sqrt{u_1^2 + a^2} - \frac{u_0}{a^2}\sqrt{u_0^2 + a^2}\right)$$

Since $\sqrt{u_0^2 + a^2} = |\Delta S_0|$, $\sqrt{u_1^2 + a^2} = |\Delta S_1|$, and
$\dfrac{u_0}{\sqrt{\Delta^2\alpha_0}} = \dfrac{\Delta S_0.\Delta^2 S_0}{|\Delta^2 S_0|^2}$, $\dfrac{u_1}{\sqrt{\Delta^2\alpha_0}} = \dfrac{\Delta S_1.\Delta^2 S_0}{|\Delta^2 S_0|^2}$,
and $\dfrac{a^2}{\sqrt{\Delta^2\alpha_0}} = \dfrac{|\Delta S_0|^2|\Delta S_1|^2 - (\Delta S_0.\Delta S_1)^2}{|\Delta^2 S_0|^3}$, we obtain the desired result.
□

Proposition 9 *The two sequences* (L^n) *and* (l^n) *converge linearly to L. More specifically :*

$$\lim_{n\to+\infty} \frac{L^{n+1} - L}{L^n - L} = \lim_{n\to+\infty} \frac{l^{n+1} - L}{l^n - L} = \lim_{n\to+\infty} \frac{L^{n+1} - l^{n+1}}{L^n - l^n} = \frac{1}{4}$$

Proof : For any function $g \in C^\infty[0,1]$, there hold respectively the two following asymptotic expansions for the values of the trapezoidal rule and of the midpoint rule with step length $1/2^n$, (see e.g. [2], p. 189-190):

$$\int_0^1 g(t)dt \quad - \quad \frac{1}{2^n}\left(\frac{1}{2}g(0) + \sum_{i=1}^{2^n-1} g\left(\frac{i}{2^n}\right) + \frac{1}{2}g(1)\right) = -\frac{1}{12}\frac{g'(1) - g'(0)}{4^n} + O(4^{-2n})$$

$$\int_0^1 g(t)dt \quad - \quad \frac{1}{2^n}\sum_{i=1}^{2^n} g(\frac{2i-1}{2^{n+1}}) = \frac{1}{24}\frac{g'(1)-g'(0)}{4^n} + O(4^{-2n})$$

This proves that the two sequences converge linearly to the integral with a convergence rate equal to 1/4.

Applying these results to $g(t) = |M'(t)|$ gives the two first announced limits, since
$L = \int_0^1 |M'(t)|dt,$
$L^n = \frac{1}{2^n}\Big(\frac{1}{2}|M'(0)| + \sum_{i=1}^{2^n-1} |M'(\frac{i}{2^n})| + \frac{1}{2}|M'(1)|\Big)$ and
$l^n = \frac{1}{2^n}\Big(\sum_{i=1}^{2^n} |M'(\frac{2i-1}{2^{n+1}})|\Big).$
It suffices to prove these equalities for $n = 0$, as the general formulas are summations of local formulas in each interval $[\frac{i}{2^n}, \frac{i+1}{2^n}], i \in \{0, \ldots, 2^n - 1\}$.
With the notations of the previous proposition, we easily compute:
$L^0 = |\Delta S_0| + |\Delta S_1|$ and $l^0 = |S_0 S_2| = |\Delta S_0 + \Delta S_1|$.
Since $|M'(0)| = 2|\Delta S_0|$, $|M'(1)| = 2|\Delta S_1|$ and
$|M'(\frac{1}{2}) = |\Delta S_0 + \Delta S_1|$, we see immediately that
$L^0 = \frac{1}{2}(|M'(0)| + |M'(1)|)$ and $l^0 = |M'(\frac{1}{2})|$
are respectively the values of the trapezoidal rule and of the midpoint rule applied to the integral $L = \int_0^1 |M'(t)|dt$. Similarly, L^n and l^n are respectively the values of the composed trapezoidal and midpoint rules, with a step of length $h = \frac{1}{2^n}$, applied to the same integral.
Finally, we compute

$$g'(0) = 2\frac{\Delta S_0.\Delta^2 S_0}{|\Delta S_0|} = 2|\Delta^2 S_0|\cos\theta_0$$

$$g'(1) = 2\frac{\Delta S_1.\Delta^2 S_0}{|\Delta S_1|} = 2|\Delta^2 S_0|\cos\theta_1$$

where $\theta_0 = \angle(\Delta S_0, \Delta^2 S_0)$ and $\theta_1 = \angle(\Delta S_1, \Delta^2 S_0)$,
hence $g'(1) - g'(0) = 2|\Delta^2 S_0|(\cos\theta_1 - \cos\theta_0) \neq 0$.
Similarly, the first term of the asymptotic expansion of $L^n - l^n$ being equal to $\frac{1}{8}\frac{g'(1)-g'(0)}{4^n}$, we deduce the third limit in the same way. □

6. Examples

In an orthonormal basis, let $A = (-1, 1)$, $B = (0, 0)$ and $C = (1, 2)$; the following three tables show the computed values of the successive lengths L^n, l^n and the ratios of their differences for different values of γ.

n	L^n	l^n	$\frac{L^n - L^{n-1}}{L^{n-1} - L^{n-2}}$	$\frac{l^n - l^{n-1}}{l^{n-1} - l^{n-2}}$
0	1.825141	1.118034		
1	1.421080	1.264372		
2	1.331533	1.294985	0.221618	0.209189
⋮	⋮	⋮	⋮	⋮
10	1.3038891	1.3038886	0.251342	0.251176
11	1.3038888	1.3038887	0.251460	0.251360

$\gamma = 1.5$,

n	L^n	l^n	$\frac{L^n - L^{n-1}}{L^{n-1} - L^{n-2}}$	$\frac{l^n - l^{n-1}}{l^{n-1} - l^{n-2}}$
0	1.825141	1.118034		
1	1.471587	1.315779		
2	1.393683	1.355960	0.220347	0.203198
⋮	⋮	⋮	⋮	⋮
10	1.3685601	1.3685596	0.250000	0.250000
11	1.3685599	1.3685597	0.250000	0.250000

$\gamma = 2$,

n	L^n	l^n	$\frac{L^n - L^{n-1}}{L^{n-1} - L^{n-2}}$	$\frac{l^n - l^{n-1}}{l^{n-1} - l^{n-2}}$
0	1.825141	1.118034		
1	1.542298	1.398749		
2	1.484878	1.447380	0.203009	0.173241
⋮	⋮	⋮	⋮	⋮
10	1.4650072	1.4650067	0.248206	0.248205
11	1.4650070	1.4650069	0.248208	0.248208

$\gamma = 3$.

The last table shows the computed maximum and minimum values of $\frac{\lambda_{i+1}^n}{\lambda_i^n}$ at step $n=11$; these extrema seem to converge to $\gamma-1$ and $\dfrac{1}{\gamma-1}$ respectively for $\gamma \geq 2$ and to $\dfrac{1}{\gamma-1}$ and $\gamma-1$ for $1<\gamma\leq 2$.

$n=11$,

γ	$\max \frac{\lambda_{i+1}^n}{\lambda_i^n}$	$\min \frac{\lambda_{i+1}^n}{\lambda_i^n}$
1.5	1.986	0.506
1.8	1.251	0.801
2	1.001	0.999
2.5	1.500	0.667
3	2.000	0.500
5	4.000	0.250
10	9.000	0.111

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