

de Rham Transform of a Hermite Subdivision Scheme

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Abstract

For a Hermite subdivision scheme \mathcal{H} of degree d , we define a spectral condition. We have already proved that if it is fulfilled, then there exists an associated affine subdivision scheme \mathcal{S} . Moreover, if T is the subdivision matrix of the first difference process $\Delta\mathcal{S}$, if the spectral radius $\rho(T)$ is less than 1, then \mathcal{S} is C^0 and \mathcal{H} is C^d . Generalizing the de Rham corner cutting, from every Hermite subdivision scheme \mathcal{H} , we build a new Hermite subdivision scheme, the de Rham transform $\tilde{\mathcal{H}}$. If \mathcal{H} satisfies the spectral condition, then its de Rham transform fulfils it. If $\tilde{\mathcal{S}}$ is the associated subdivision scheme to $\tilde{\mathcal{H}}$ and if T and \tilde{T} are the respective subdivision matrices corresponding to $\Delta\mathcal{S}$ and $\Delta\tilde{\mathcal{S}}$, then the spectral radii $\rho(T)$ and $\rho(\tilde{T})$ allow the comparison between the indices of smoothness of the limit functions of the schemes \mathcal{H} and $\tilde{\mathcal{H}}$. We apply these results to the Merrien class of Hermite subdivision schemes of degree 1.

1 Introduction

A *Hermite subdivision scheme* \mathcal{H} of degree d is a recursive scheme for computing a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and its d derivatives $\phi', \dots, \phi^{(d)}$. The initial state of the scheme is a vector function $f_0 : \mathbb{Z} \rightarrow \mathbb{R}^{d+1}$. The first component of f_0 is a control value for ϕ , the second component, for ϕ' and so on. The sequence of *refinements* $f_n : \mathbb{Z} \rightarrow \mathbb{R}^{d+1}$, $n > 0$, is recursively defined through a family of $(d+1) \times (d+1)$ matrices $\{A(\alpha) = (a_{ij}(\alpha))_{i,j=0,\dots,d}\}_{\alpha \in \mathbb{Z}}$, a finite number of them being non-zero, by

$$D^{n+1}f_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} A(\alpha - 2\beta)D^n f_n(\beta), \quad \alpha \in \mathbb{Z}, n \geq 0, \quad (1)$$

where D is the diagonal matrix with diagonal elements $1, 1/2, \dots, 1/2^d$.

The family of matrices $\{A(\alpha)\}_{\alpha \in \mathbb{Z}}$ is called the *mask* of the Hermite subdivision scheme \mathcal{H} . The *support* of \mathcal{H} is the smallest interval $[\sigma, \sigma']$ containing $\{\alpha \in \mathbb{Z} : A(\alpha) \neq 0\}$.

The outline of the paper is the following: in Section 2, we recall the main results of the previous paper [4] and we define what the spectral condition for a Hermite subdivision scheme \mathcal{H} is. Under this condition, we associate with \mathcal{H} a scalar affine subdivision scheme \mathcal{S} . If T is the subdivision matrix of the first difference process $\Delta\mathcal{S}$ and if the spectral radius $\rho(T)$ is less than 1, then \mathcal{S} is C^0 and \mathcal{H} is C^d as shown in [4]. In Section 3, generalizing from de Rham corner cutting [2] and Chaikin algorithm [1], from every Hermite subdivision scheme \mathcal{H} , we build a new Hermite subdivision scheme, de Rham transform $\tilde{\mathcal{H}}$. If \mathcal{H} satisfies the spectral condition, then its de Rham transform fulfils it, which insures the existence of the associated scalar affine subdivision scheme $\tilde{\mathcal{S}}$ and of the subdivision matrix \tilde{T} of $\Delta\tilde{\mathcal{S}}$. In Section 3, we compare the indices of smoothness of the limit functions of the schemes \mathcal{H} and $\tilde{\mathcal{H}}$ with the help of the spectral radii $\rho(T)$ and $\rho(\tilde{T})$. In particular, we complete the computation of T and \tilde{T} in the case of any Hermite subdivision scheme belonging to the Merrien class as defined in [7].

2 Associated Vector Subdivision Scheme and Convergence

Definition 1 *We say that a Hermite subdivision scheme of degree d is C^d if for every sequence of refinements $f_n : \mathbb{Z} \rightarrow \mathbb{R}^{d+1}$, there exists a C^d -function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for which for every $\epsilon > 0$ and for every $L > 0$, there exists N such that for every $n > N$, for every $\alpha \in [-L2^n, L2^n]$, $|f_n^{(i)}(\alpha) - \phi^{(i)}(\alpha/2^n)| < \epsilon$, for $i = 0, \dots, d$. The function ϕ is called the limit function associated to the refinements f_n .*

Definition 2 *A Hermite subdivision scheme of degree d satisfies the spectral condition if for $k = 0, \dots, d$, there exists a polynomial p_k of degree k such that for all $\alpha \in \mathbb{Z}$*

$$\sum_{\beta \in \mathbb{Z}} A(\alpha - 2\beta)v_k(\beta) = v_k(\alpha)/2^k \text{ where } v_k(\alpha) = (p_k(\alpha), \dots, p_k^{(d)}(\alpha))^T. \quad (2)$$

We use the notation \mathcal{P}_n for the space of polynomials of degree inferior or equal to n .

Lemma 1 *For a given Hermite subdivision scheme of degree d satisfying the spectral condition (2) with polynomials p_0, p_1, \dots, p_d , each polynomial p_k of degree k is unique up to a multiplicative factor.*

Proof: Let V the set of vector functions $v : \mathbb{Z} \rightarrow \mathbb{C}^{d+1}$ for which there exists a polynomial $p \in \mathcal{P}_d$ such that $v(\alpha) = (p(\alpha), p'(\alpha), \dots, p^{(d)}(\alpha))^T$ for every $\alpha \in \mathbb{Z}$. The dimension of V is $d + 1$: the set $\{v_0, v_1, \dots, v_d\}$ is linearly independent as a set of eigenvectors corresponding to distinct eigenvalues, thus it is a basis of V . Moreover, the polynomials p_0, p_1, \dots, p_d are linearly independent since their degrees are distinct.

Let $q \in \mathcal{P}_d$ be such that $w(\alpha) = (q(\alpha), q'(\alpha), \dots, q^{(d)}(\alpha))^T$ is an eigenvector corresponding to the value $1/2^k$, then $q = \sum_{i=0}^d c_i p_i$ and $w = \sum_{i=0}^d c_i v_i$. If $Hw(\alpha) = \sum_{\beta \in \mathbb{Z}} A(\alpha - 2\beta)w(\beta)$, then $w = 2^k Hw$ and $\sum_{i=0}^d 2^{k-i} c_i v_i = \sum_{i=0}^d c_i v_i$. If $i \neq k$, then $c_i = 0$ and $w = c_k v_k$. In particular, $q = c_k p_k$. \square

Definition 3 A Hermite subdivision scheme is interpolatory if $A(0) = D$ and for all $\alpha \in \mathbb{Z}$ with $\alpha \neq 0$, $A(2\alpha) = 0$. In this case, for $\alpha \in \mathbb{Z}$, $f_n(\alpha) = f_{n+1}(2\alpha)$.

Lemma 2 Let \mathcal{H} be an interpolatory Hermite subdivision scheme of degree d which satisfies the spectral condition (2) for a polynomial p_k of degree k , $k \in [0, d]$, then $p_k(x)$ is a multiple of x^k .

Proof: We choose the value $\alpha = 0$ in (2) and we get $Dv_k(0) = v_k(0)/2^k$, i.e. $p_k^{(i)}(0)/2^i = p_k^{(i)}(0)/2^k$ for $i = 0, 1, \dots, d$, then $p_k^{(i)}(0) = 0$ for $i \neq k$. From that it follows that $p_k(x) = \sum_{i=0}^d p_k^{(i)}(0)x^i/i! = p_k^{(k)}(0)x^k/k!$. \square

A vector subdivision matrix of order p is a matrix function $S : \mathbb{Z}^2 \rightarrow \mathbb{R}^{p \times p}$ for which there exists an interval $[\sigma, \sigma']$, $\sigma, \sigma' \in \mathbb{Z}$, $\sigma \leq \sigma'$ such that $S(\alpha, \beta) = 0$ whenever $\alpha - 2\beta \notin [\sigma, \sigma']$. If $g_0 : \mathbb{R}^p \rightarrow \mathbb{R}$, then the recursive formula $g_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} S(\alpha, \beta)g_n(\beta)$ generates the refinements of g_0 and defines a vector subdivision scheme \mathcal{S} of order p . The interval $[\sigma, \sigma']$ is a support of \mathcal{S} . If there exists a matrix function $B : \mathbb{Z} \rightarrow \mathbb{R}^{p \times p}$ such that $S(\alpha, \beta) = B(\alpha - 2\beta)$, then B is the mask of \mathcal{S} . S (and its corresponding vector subdivision scheme) is affine if $\sum_{\beta \in \mathbb{Z}} \sum_{j=1}^p s_{ij}(\alpha, \beta) = 1$, $\alpha \in \mathbb{Z}$, $i = 1, 2, \dots, p$, where $s_{ij}(\alpha, \beta)$ are the entries of $S(\alpha, \beta)$. The subdivision scheme is scalar if $p = 1$. We say that a vector subdivision scheme \mathcal{S} of order p is C^0 if for every sequence of refinements $f_n : \mathbb{Z} \rightarrow \mathbb{R}^p$ of \mathcal{S} , there is a continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}^p$ such that for every $\epsilon > 0$ and for every $L > 0$, there exists N such that $\|f_n(\alpha) - \phi(\alpha/2^n)\| < \epsilon$ for every $n > N$ and every $\alpha \in [-L2^n, L2^n]$. The function ϕ is called the limit function associated with the refinements f_n . If for any limit function, all of its components are the same, we say that \mathcal{S} is C^0 with equal components.

For $f = (f^{(0)}, \dots, f^{(d)})^T$ where $f^{(i)} : \mathbb{Z} \rightarrow \mathbb{C}$, $i = 0, \dots, d$, if P_0, P_1 are the

respective square matrices of order $d + 1$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{(d-2)!}{(d-3)!} & 0 \\ 0 & -\frac{(d-1)!}{0!} & \cdots & -\frac{(d-2)!}{(d-1)!} & 0 \\ -\frac{d!}{0!} & -\frac{d!}{1!} & \cdots & -\frac{d!}{(d-1)!} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1! & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & (d-1)! & \cdots & 0 & 0 \\ d! & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3)$$

we define the operator Pf

$$Pf(\alpha) = P_0f(\alpha) + P_1f(\alpha + 1), \alpha \in \mathbb{Z}. \quad (4)$$

Theorem 3 *Let \mathcal{H} be a Hermite subdivision scheme of degree d which satisfies the spectral condition and whose mask $A(\alpha) = (a_{ij}(\alpha))_{i,j=0,\dots,d}$ has support included in $[\sigma, \sigma']$. Let $f_n = (f_n^{(0)}, f_n^{(1)}, \dots, f_n^{(d)})^T$, $n = 0, 1, 2, \dots$ be the refinements of a \mathcal{H} . We set $g_n = 2^{nd}PD^n f_n$ and $C(\alpha) = 2^d(P_0A(\alpha) + P_1A(\alpha + 1))$ where P, P_0, P_1 are defined in (3-4). If we define the matrix function $B(\alpha) = (b_{ij}(\alpha))_{i,j=0,\dots,d}$ by*

$$b_{i,d-j}(\alpha) = \frac{1}{(d-j)!} \sum_{\beta=1}^{\infty} \sum_{k=0}^j c_{i,k}(\alpha - 2\beta) \frac{(\beta-1)^{j-k}}{(j-k)!}, \quad j = 0, \dots, d-1 \quad (5)$$

$$b_{i,0}(\alpha) = c_{id}(\alpha) \quad (6)$$

for $i = 0, \dots, d$, then the support of B is contained in $[\sigma - 1, \sigma']$ and the sequence g_n is the sequence of refinements of an affine vector subdivision scheme \mathcal{S} whose mask is B .

See [4] for the proof of the previous and the following theorems.

Theorem 4 *Let \mathcal{H} be a Hermite subdivision scheme of degree d which satisfies the spectral condition. Let $B(\alpha) = (b_{ij}(\alpha))_{i,j=0,1,\dots,d}$ be the mask of the Taylor subdivision scheme \mathcal{S} associated with \mathcal{H} . We define the functions $s, t : \mathbb{Z}^2 \rightarrow \mathbb{R}$*

$$s((d+1)\alpha + i, (d+1)\beta + j) = b_{ij}(\alpha - 2\beta), \quad \alpha, \beta \in \mathbb{Z}, i, j = 0, 1, \dots, d, \quad (7)$$

$$t(\alpha, \beta) = - \sum_{\gamma=-\infty}^{\beta} [s(\alpha + 1, \gamma) - s(\alpha, \gamma)] = \sum_{\gamma=\beta+1}^{\infty} [s(\alpha + 1, \gamma) - s(\alpha, \gamma)]. \quad (8)$$

If, moreover, there is an integer n such that

$$\max_{\beta \in \mathbb{Z}} \left\{ \sum_{\alpha \in [0, (d+1)2^n - 1]} |t_n(\alpha, \beta)| \right\} < 1 \quad (9)$$

where $(t_n(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$ is the n -th power of the matrix $T = (t(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$, then the Taylor subdivision scheme \mathcal{S} is C^0 with equal components and the Hermite subdivision scheme \mathcal{H} is C^d .

Remark 1 Condition (9) means that $\|T^n\|_\infty < 1$. As the spectral radius of T is $\rho(T) = \inf\{(\|T^n\|_\infty)^{1/n} : n \in \mathbb{N}\}$, this condition is equivalent to $\rho(T) < 1$.

3 de Rham Transform

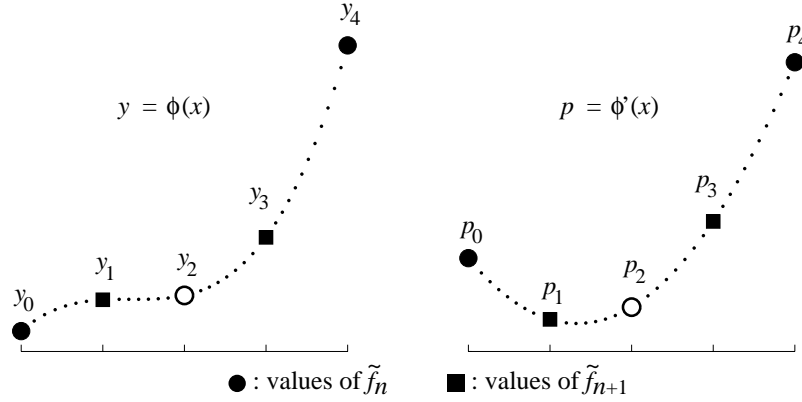
For a Hermite subdivision scheme \mathcal{H} of degree d , with mask $\{A(\alpha)\}_{\alpha \in \mathbb{Z}}$, let us define a new Hermite subdivision scheme $\tilde{\mathcal{H}}$. From the initial state of the scheme $\tilde{f}_0 : \mathbb{Z} \rightarrow \mathbb{R}^{d+1}$, we define the sequence \tilde{f}_n for $n > 0$ by

$$D^{n+1}g(\beta) = \sum_{\gamma \in \mathbb{Z}} A(\beta - 2\gamma) D^n \tilde{f}_n(\gamma), \beta \in \mathbb{Z} \quad (10)$$

$$D^{n+2}h(\alpha) = \sum_{\beta \in \mathbb{Z}} A(\alpha - 2\beta) D^{n+1}g(\beta), \alpha \in \mathbb{Z} \quad (11)$$

$$\tilde{f}_{n+1}(\alpha) = h(2\alpha + 1), \alpha \in \mathbb{Z}. \quad (12)$$

(See Figure 1.)



$$\begin{aligned} g(2\alpha) &= h(4\alpha) = (y_0, p_0) = \tilde{f}_n(\alpha), \\ h(4\alpha+1) &= (y_1, p_1) = \tilde{f}_{n+1}(2\alpha), \\ g(2\alpha+1) &= h(4\alpha+2) = (y_2, p_2), \\ h(4\alpha+3) &= (y_3, p_3) = \tilde{f}_{n+1}(2\alpha+1), \\ g(2\alpha+2) &= h(4\alpha+4) = (y_4, p_4) = \tilde{f}_n(\alpha+1). \end{aligned}$$

Figure 1: For $d = 1$, the construction of \tilde{f}_{n+1} when \mathcal{H} generates the Hermite cubic interpolant.

Then

$$\begin{aligned}
& D^{n+1}\tilde{f}_{n+1}(\alpha) = D^{-1}D^{n+2}h(2\alpha + 1) \\
&= D^{-1}\sum_{\beta\in\mathbb{Z}}A(2\alpha + 1 - 2\beta)\sum_{\gamma\in\mathbb{Z}}A(\beta - 2\gamma)D^n\tilde{f}_n(\gamma) \\
&= D^{-1}\sum_{\gamma\in\mathbb{Z}}\left(\sum_{\beta\in\mathbb{Z}}A(2\alpha + 1 - 2\beta)A(\beta - 2\gamma)\right)D^n\tilde{f}_n(\gamma) \\
&= D^{-1}\sum_{\gamma\in\mathbb{Z}}\left(\sum_{\beta\in\mathbb{Z}}A(2\alpha + 1 - 4\gamma - 2\beta)A(\beta)\right)D^n\tilde{f}_n(\gamma) \\
&= \sum_{\gamma\in\mathbb{Z}}\tilde{A}(\alpha - 2\gamma)D^n\tilde{f}_n(\gamma),
\end{aligned}$$

where

$$\tilde{A}(\alpha) = D^{-1}\sum_{\beta\in\mathbb{Z}}A(2\alpha + 1 - 2\beta)A(\beta), \alpha \in \mathbb{Z}. \quad (13)$$

Definition 4 Let \mathcal{H} be a Hermite subdivision scheme with mask $A(\alpha)$. The de Rham transform $\tilde{\mathcal{H}}$ of \mathcal{H} is the Hermite subdivision scheme whose mask $\tilde{A}(\alpha)$ is defined by (13).

If the support of \mathcal{H} is $[\sigma, \sigma']$, then the support of its de Rham transform $\tilde{\mathcal{H}}$ is contained in $[(3\sigma - 1)/2, (3\sigma' - 1)/2]$.

Theorem 5 Let \mathcal{H} be a Hermite subdivision scheme of degree d which satisfies the spectral condition (2), then the de Rham transform $\tilde{\mathcal{H}}$ satisfies the corresponding spectral condition $\sum_{\beta\in\mathbb{Z}}\tilde{A}(\alpha - 2\beta)\tilde{v}_k(\beta) = \tilde{v}_k(\alpha)/2^k$ where $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha), \tilde{p}'_k(\alpha), \dots, \tilde{p}_k^{(d)}(\alpha))^T$ for an appropriate sequence of polynomials \tilde{p}_k of degree k , $k = 0, \dots, d$.

Proof: If we use the notations of Definition 2, for $i = 0, \dots, d$, let $v_i(\alpha) = (p_i(\alpha), \dots, p_i^{(d)}(\alpha))^T$ be the eigenvector of $H = (A(\alpha - 2\beta))_{\alpha, \beta \in \mathbb{Z}}$ for the eigenvalue $1/2^i$ where p_i is a polynomial of degree i with its leading coefficient equal to $1/i!$. For $\tilde{H} = (\tilde{A}(\alpha - 2\beta))_{\alpha, \beta \in \mathbb{Z}}$, we will build the polynomials \tilde{p}_k of degree k with leading coefficient equal to $1/k!$ and the corresponding eigenvector $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha), \dots, \tilde{p}_k^{(d)}(\alpha))^T$ by a finite recursion.

We begin with a computation to prove that:

$$\sum_{\beta\in\mathbb{Z}}\tilde{A}(\alpha - 2\beta)v_k(\beta) = 1/4^k D^{-1}v_k(2\alpha + 1), \quad k = 0, \dots, d \quad (14)$$

$$\begin{aligned}
& \sum_{\beta \in \mathbb{Z}} \tilde{A}(\alpha - 2\beta)v_k(\beta) = D^{-1} \sum_{\beta \in \mathbb{Z}} \sum_{\gamma \in \mathbb{Z}} A(2\alpha - 4\beta - 2\gamma + 1)A(\gamma)v_k(\beta) \\
&= D^{-1} \sum_{\beta \in \mathbb{Z}} \sum_{\gamma' \in \mathbb{Z}} A(2\alpha + 1 - 2\gamma')A(\gamma' - 2\beta)v_k(\beta) \\
&= D^{-1} \sum_{\gamma' \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} A(2\alpha + 1 - 2\gamma')A(\gamma' - 2\beta)v_k(\beta) \\
&= D^{-1} \sum_{\gamma' \in \mathbb{Z}} A(2\alpha + 1 - 2\gamma')1/2^k v_k(\gamma') = 1/4^k D^{-1}v_k(2\alpha + 1).
\end{aligned}$$

Step 0. Let $\tilde{p}_0(x) = 1 = p_0(x)$. If $\tilde{v}_0(\alpha) = (\tilde{p}_0(\alpha), \dots, \tilde{p}_0^{(d)}(\alpha))^T$, then $v_0(\alpha) = \tilde{v}_0(\alpha) = (1, 0, \dots, 0)^T$. With (14), we get

$$\sum_{\beta \in \mathbb{Z}} \tilde{A}(\alpha - 2\beta)\tilde{v}_0(\beta) = D^{-1}v_0(2\alpha + 1) = v_0(\alpha) = \tilde{v}_0(\alpha)$$

and $\tilde{v}_0(\alpha)$ is an eigenvector of \tilde{H} for the eigenvalue 1.

Step k with $0 < k \leq d$. Let us suppose that for $i = 0, \dots, k-1$, $\tilde{v}_i(\alpha) = (\tilde{p}_i(\alpha), \dots, \tilde{p}_i^{(d)}(\alpha))^T$ is an eigenvector of $\tilde{H} = (\tilde{A}(\alpha - 2\beta))_{\alpha, \beta \in \mathbb{Z}}$ for the eigenvalue $1/2^i$ where \tilde{p}_i is a polynomial of degree i with its leading coefficient equal to $1/i!$

Since $\{p_k, \tilde{p}_{k-1}, \dots, \tilde{p}_0\}$ is a basis of \mathcal{P}_k , and since we know that the leading coefficient of p_k is $1/k!$, we can write $p_k(2x + 1) = 2^k p_k(x) + \sum_{i=0}^{k-1} \lambda_{k,i} \tilde{p}_i(x)$. When we differentiate up to d times, both sides of the previous equation with respect to x , we get:

$$D^{-1}v_k(2\alpha + 1) = 2^k v_k(\alpha) + \sum_{i=0}^{k-1} \lambda_{k,i} \tilde{v}_i(\alpha), \quad \alpha \in \mathbb{Z}. \quad (15)$$

Let us define $\tilde{p}_k(x) = p_k(x) + \sum_{i=0}^{k-1} \mu_{k,i} \tilde{p}_i(x)$ where $\mu_{k,i} = -\lambda_{k,i} \frac{2^{i-k}}{2^{k-2i}}$ and for $\alpha \in \mathbb{Z}$, let us set $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha), \dots, \tilde{p}_k^{(d)}(\alpha))^T$. Firstly, \tilde{p}_k is a polynomial of degree k and its leading coefficient is exactly $1/k!$. Secondly, when we differentiate \tilde{p}_k , we obtain:

$$\tilde{v}_k(\alpha) = v_k(\alpha) + \sum_{i=0}^{k-1} \mu_{k,i} \tilde{v}_i(\alpha). \quad (16)$$

Thirdly, for $i = 0, \dots, k-1$, we already know that $\tilde{v}_i(\alpha)$ is an eigenvector of \tilde{H} for the eigenvalue $1/2^i$.

If we set $w_k(\alpha) = \sum_{\beta \in \mathbb{Z}} \tilde{A}(\alpha - 2\beta) \tilde{v}_k(\beta)$, with (14), we get:

$$\begin{aligned}
w_k(\alpha) &= 1/4^k D^{-1} v_k(2\alpha + 1) + \sum_{i=0}^{k-1} \mu_{k,i} / 2^i \tilde{v}_i(\alpha) \\
&= 1/2^k v_k(\alpha) + \sum_{i=0}^{k-1} (\lambda_{k,i} / 4^k + \mu_{k,i} / 2^i) \tilde{v}_i(\alpha) \quad (\text{from (15)}) \\
&= 1/2^k v_k(\alpha) + \sum_{i=0}^{k-1} \mu_{k,i} \left(-\frac{1}{4^k} \frac{2^k - 2^i}{2^{i-k}} + \frac{1}{2^i} \right) \tilde{v}_i(\alpha) \\
&= 1/2^k \left(v_k(\alpha) + \sum_{i=0}^{k-1} \mu_{k,i} \tilde{v}_i(\alpha) \right) = 1/2^k \tilde{v}_k(\alpha) \quad (\text{from (16)}).
\end{aligned}$$

So that $\tilde{v}_k(\alpha)$ is an eigenvector of \tilde{H} for the eigenvalue $1/2^k$. The spectral condition has been proved by induction. \square

Corollary 6 *Let \mathcal{H} be an interpolatory scheme of degree d which satisfies the spectral condition, then its de Rham transform $\tilde{\mathcal{H}}$ satisfies the corresponding spectral condition with the sequence of polynomials $\tilde{p}_k(x) = (x - 1/2)^k / k!$, $k = 0, \dots, d$.*

Proof: We use the notations of the previous theorem. From Lemma 1, \mathcal{H} satisfies the spectral condition (2) with the sequence $p_k(x) = x^k / k!$.

For $k = 0$, $p_0(x) = 1$, and we have already seen that $\tilde{p}_0(x) = 1$.

Let us suppose that for $i = 0, \dots, k-1$, $\tilde{p}_i(x) = (x - 1/2)^i / i!$, then

$$\begin{aligned}
p_k(2x + 1) &= 2^k x^k / k! + 2^k (x - 1/2 + 1)^k / k! - 2^k (x - 1/2 + 1/2)^k / k! \\
&= 2^k x^k / k! + \frac{2^k}{k!} \sum_{i=0}^{k-1} \binom{k}{i} (1 - 1/2^{k-i}) (x - 1/2)^i \\
&= 2^k p_k(x) + \sum_{i=0}^{k-1} \lambda_{k,i} \tilde{p}_i(x), \quad \text{with } \lambda_{k,i} = \frac{2^k - 2^i}{(k-i)!}.
\end{aligned}$$

Since $\mu_{k,i} = -\lambda_{k,i} \frac{2^{i-k}}{2^{k-2^i}} = -\frac{2^{i-k}}{(k-i)!}$, we obtain

$$\begin{aligned}
\tilde{p}_k(x) &= p_k(x) + \sum_{i=0}^{k-1} \mu_{k,i} \tilde{p}_i(x) = \frac{(x - \frac{1}{2} + \frac{1}{2})^k}{k!} - \sum_{i=0}^{k-1} \frac{2^{i-k}}{(k-i)!} \tilde{p}_i(x) \\
&= \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} 2^{i-k} (x - 1/2)^i - \frac{1}{k!} \sum_{i=0}^{k-1} \binom{k}{i} (x - 1/2)^i 2^{i-k} \\
&= (x - 1/2)^k / k!
\end{aligned}$$

and the result is proved by this finite induction. \square

4 Examples of Hermite Schemes and their de Rham Transforms

As a model example, we recall the interpolatory Hermite subdivision scheme $\mathcal{H}(\lambda, \mu)$ proposed by Merrien [7]. The non-zero matrices of its mask of are

$$A(-1) = \frac{1}{4} \begin{pmatrix} 2 & 4\lambda \\ 2\mu & 1 - \mu \end{pmatrix}, A(0) = D, A(1) = \frac{1}{4} \begin{pmatrix} 2 & -4\lambda \\ -2\mu & 1 - \mu \end{pmatrix}.$$

For every choice of λ and μ , $\mathcal{H}(\lambda, \mu)$ reproduces affine functions and satisfies the spectral condition (2) with $p_0(x) = 1$, $p_1(x) = x$. According to Theorem 3, an affine vector subdivision scheme \mathcal{S} is associated with $\mathcal{H}(\lambda, \mu)$. The nonzero matrices $B(\alpha)$ of the mask of \mathcal{S} are $B(\alpha)$, $\alpha = -2, \dots, 1$, i.e. $\begin{pmatrix} 0 & 0 \\ 2\lambda & 0 \end{pmatrix}$, $\begin{pmatrix} (1 - \mu)/2 & 0 \\ -2\lambda & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -2\lambda & 1 \end{pmatrix}$, $\begin{pmatrix} (1 - \mu)/2 & \mu \\ 2\lambda & 1 \end{pmatrix}$.

The subdivision matrix $T = (t(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$ of the difference subdivision scheme $\Delta\mathcal{S}$ can be written as follows.

$$(t(\alpha, \beta))_{\alpha \in [0, 3], \beta = 0, 1} = \frac{1}{2} \begin{pmatrix} 2 + 4\lambda & 4\lambda \\ -1 - 4\lambda + \mu & 1 - 4\lambda - \mu \\ 1 - 4\lambda - \mu & -1 - 4\lambda + \mu \\ 4\lambda & 2 + 4\lambda \end{pmatrix}.$$

For every $\alpha \in [0, 3]$, $t(\alpha, \beta) = 0$ if $\beta \notin [0, 1]$. $t(\alpha + 4\gamma, \beta + 2\gamma) = t(\alpha, \beta)$.

The support of de Rham transform $\tilde{\mathcal{H}}(\lambda, \mu)$ is $[-2, 1]$. The respective matrices $\tilde{A}(-2)$, $\tilde{A}(-1)$, $\tilde{A}(0)$, $\tilde{A}(1)$ of its mask are

$$\begin{aligned} & \frac{1}{8} \begin{pmatrix} 2 + 4\lambda\mu & 6\lambda - 2\lambda\mu \\ 6\mu - 2\mu^2 & (1 - \mu)^2 + 8\lambda\mu \end{pmatrix}, \frac{1}{8} \begin{pmatrix} 6 - 4\lambda\mu & 6\lambda + 2\lambda\mu \\ 6\mu - 2\mu^2 & 3 - 4\mu + \mu^2 - 8\lambda\mu \end{pmatrix}, \\ & \frac{1}{8} \begin{pmatrix} 6 - 4\lambda\mu & -6\lambda - 2\lambda\mu \\ -6\mu + 2\mu^2 & 3 - 4\mu + \mu^2 - 8\lambda\mu \end{pmatrix}, \frac{1}{8} \begin{pmatrix} 2 + 4\lambda\mu & -6\lambda + 2\lambda\mu \\ -6\mu + 2\mu^2 & (1 - \mu)^2 + 8\lambda\mu \end{pmatrix}. \end{aligned}$$

$\tilde{\mathcal{H}}(\lambda, \mu)$ satisfies the spectral condition with $\tilde{p}_0(x) = 1$, $\tilde{p}_1(x) = x - 1/2$. From Theorem 3, an affine vector subdivision scheme $\tilde{\mathcal{S}}$ is associated with $\tilde{\mathcal{H}}(\lambda, \mu)$. The nonzero matrices $\tilde{B}(\alpha)$ of the mask of $\tilde{\mathcal{S}}$ are $\tilde{B}(\alpha)$, $\alpha \in [-3, 1]$. Using Theorem 4, we define two infinite matrices $\tilde{S} = (\tilde{s}(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$ and $\tilde{T} = (\tilde{t}(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$. For $\alpha, \beta \in \mathbb{Z}$, for $i, j = 0, 1$, $\tilde{s}(2\alpha + i, 2\beta + j) = \tilde{b}_{ij}(\alpha, \beta)$ and $\tilde{t}(\alpha, \beta) = -\sum_{\gamma=-\infty}^{\beta} [\tilde{s}(\alpha + 1, \gamma) - \tilde{s}(\alpha, \gamma)]$.

After computation, we obtain the following information about \tilde{T} . The entries

$\tilde{t}(\alpha, \beta)$ are 0 if $\alpha = [0, 3]$ and $\beta \notin [0, 3]$. The matrix $(\tilde{t}(\alpha, \beta))_{\alpha, \beta \in [0, 3]}$ is

$$\frac{1}{4} \begin{pmatrix} w_1 & w_2 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ w_3 & w_4 & w_5 & w_6 \\ w_6 & w_5 & w_4 & w_3 \end{pmatrix}$$

where $w_1 = 3 - 4\mu + \mu^2 - 12\lambda\mu$, $w_2 = -(1 - \mu)^2 - 4\lambda\mu$, $w_3 = (1 - \mu)^2 - 6\lambda + 10\lambda\mu$, $w_4 = -1 + 4\mu - \mu^2 - 6\lambda + 6\lambda\mu$, $w_5 = 2 + 6\lambda + 2\lambda\mu$, $w_6 = 6\lambda - 2\lambda\mu$. Moreover, $\tilde{T} = (t(\alpha, \beta))$ is periodic: $\tilde{t}(\alpha + 4, \beta + 2) = t(\alpha, \beta)$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, \tilde{T} is entirely known.

Computing the spectral radii $\rho(T)$ and $\rho(\tilde{T})$ enables us to compare the two Hermite subdivision schemes, $\mathcal{H}(\lambda, \mu)$ and $\tilde{\mathcal{H}}(\lambda, \mu)$.

If $\rho(T) < 1$, then $\mathcal{H}(\lambda, \mu)$ is C^1 and any limit function ϕ of $\mathcal{H}(\lambda, \mu)$ has the property that $\phi'(x) - \phi'(x + h) = O(h^p)$ as $h \rightarrow 0$ for any $p < -\log(\rho(T))/\log 2$. We have similar properties for $\tilde{\mathcal{H}}(\lambda, \mu)$ and the Hermite subdivision scheme $\mathcal{H}(\lambda, \mu)$ generates smoother limits than $\tilde{\mathcal{H}}(\lambda, \mu)$ if $\rho(\tilde{T}) < \rho(T)$.

The numbers $\rho(T)$ and $\rho(\tilde{T})$ are not easy to compute, but they admit similar upper bounds

$$\rho(T) \leq \|T^n\|_\infty^{1/n}, \quad \rho(\tilde{T}) \leq \|\tilde{T}^n\|_\infty^{1/n} \quad (17)$$

for every $n > 0$. Lower bounds for these spectral radii follow from the following lemma.

Lemma 7 *Let $T = (t(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$ and $J \subset \mathbb{Z}$ be such that $t(\alpha, \beta) = 0$ for every $\alpha \in J$ and for every $\beta \notin J$ and let $[T]$ be the matrix $(t(\alpha, \beta))_{\alpha, \beta \in J}$, then the spectral radius of $[T]$ is bounded by the spectral radius of T .*

Proof: For any matrix $U = (u(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}}$, we set $[U] = (u(\alpha, \beta))_{\alpha, \beta \in J}$. By induction on n , we get $[T^n] = [T]^n$. It follows that $\|[T]^n\|_\infty \leq \|T^n\|_\infty$ and

$$\rho([T]) = \inf\{\|[T]^n\|_\infty^{1/n} : n > 0\} \leq \inf\{\|T^n\|_\infty^{1/n} : n > 0\} = \rho(T).$$

□

Let T be the subdivision matrix of the difference subdivision scheme $\Delta\mathcal{S}$ and let $[T]$ be the matrix $(t(\alpha, \beta))_{\alpha, \beta \in [0, 1]}$, then from Lemma 7, $\rho([T])$ is a lower bound of $\rho(T)$. Similarly, let \tilde{T} be the subdivision matrix of the difference subdivision scheme $\Delta\tilde{\mathcal{S}}$ and let $[\tilde{T}]$ be the matrix $(\tilde{t}(\alpha, \beta))_{\alpha, \beta \in [0, 3]}$, then $\rho([\tilde{T}])$ is a lower bound of $\rho(\tilde{T})$. The roots of the characteristic polynomial of $[T]$ are $w_1 \pm w_2$ and the two roots of the quadratic polynomial: $z^2 - (w_3 + w_5)z + w_3w_5 - w_4w_6$. Thus

$$\rho([\tilde{T}]) = \max(|w_1| + |w_2|, |z_1|, |z_2|)$$

where z_1, z_2 are the roots of the previous quadratic polynomial.

In the following table, we give the results of the computations of the spectral radii for 4 choices of (λ, μ) . The upper bounds for the first two choices were obtained from (17) with $n = 30$.

λ	μ	$\rho(T)$	$\rho(\tilde{T})$
1/10	3/2	1.21	$\in [0.85, 0.88]$
1/36	9/10	1.05	$\in [0.57, 0.59]$
-1/8	3/2	0.50	0.50
-1/8	3	0.50	1.75

Let $\Omega = \{(\lambda, \mu) : \mathcal{H}(\lambda, \mu) \text{ is } C^1\}$ and $\tilde{\Omega} = \{(\lambda, \mu) : \tilde{\mathcal{H}}(\lambda, \mu) \text{ is } C^1\}$. In Fig. 2, we compare Ω and $\tilde{\Omega}$. According to Theorem 18 of [3], $\Omega = \{(\lambda, \mu) : \lambda \in (-1/2, 0), \mu \in (0, \min(-1/2\lambda, 3/(1+2\lambda)))\}$; in the figure, Ω is the region bounded by the dashed curve and the segment $(-1/2, 0) \times \{0\}$. The black area in the figure is the region $\{(\lambda, \mu) : \|\tilde{T}^{20}\| < 1\}$. The set $\{(\lambda, \mu) : \rho([\tilde{T}]) < 1\}$ is the union of the black area and the gray area. There is no guarantee that every gray point belongs to $\tilde{\Omega}$. But from Theorem 4, and the previous lemma,

$$\{(\lambda, \mu) : \|\tilde{T}^{20}\| < 1\} \subset \tilde{\Omega} \subset \{(\lambda, \mu) : \rho([\tilde{T}]) < 1\}.$$

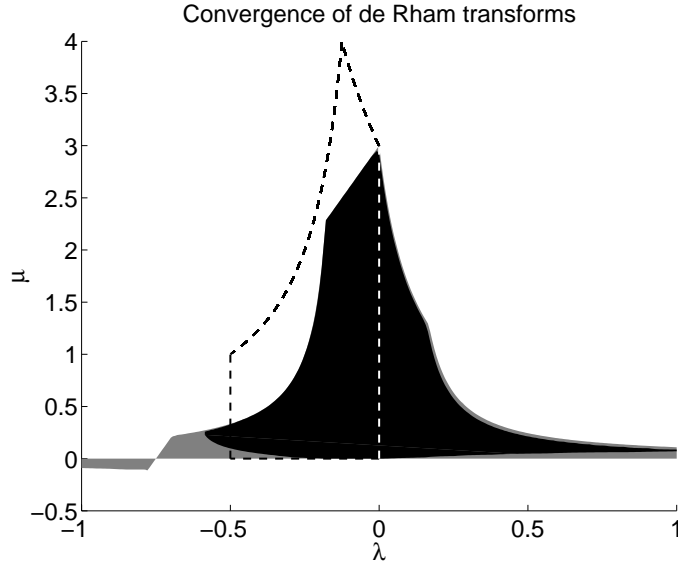


Figure 2: Comparison between the region of convergence of $\mathcal{H}(\lambda, \mu)$ and that of $\tilde{\mathcal{H}}(\lambda, \mu)$.

From Fig. 2, it is obvious that $\Omega \setminus \tilde{\Omega} \neq \emptyset$ and $\tilde{\Omega} \setminus \Omega \neq \emptyset$, which corroborates the previous table.

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