Dyadic Hermite interpolation on a rectangular mesh

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Abstract: Given f and ∇f at the vertices of a rectangular mesh, we build an interpolating function f by a subdivision algorithm. The construction on each elementary rectangle is independent of any disjoint rectangle. From the Hermite data associated with the vertices of a rectangle R, the function f is defined on a dense subset of R. Sufficient conditions are found in order to extend f to a C^1 function. Moreover infinite products and generalized radii of matrices are used to study the convergence to a C^1 function. This convergence depends on the five parameters introduced in the algorithm.

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1 Introduction

A classical method for constructing curves and surfaces in CAGD consists in binary subdivision algorithms. They are efficient tools which can be adapted

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to the computer.

Dubuc [5], Dyn et al [6], then Deslauriers et al [3, 4] have studied these methods to build interpolating curves and surfaces from Lagrange data while Merrien [10, 11] introduced the case of the Hermite interpolation. See also Dyn and Levin [7] for an analysis of general one dimensional schemes.

In this paper, given f and ∇f at the vertices of a rectangular mesh, we define an algorithm HR^1 building an interpolating C^1 function. The algorithm is local and the construction on a rectangle is independent of its neighbours. In order to get C^1 continuity across an edge, the construction depends only on the values of f and ∇f at the ends and of the length of this edge.

In Section 2, we describe the algorithm HR^1 on a single rectangle R. We build f on a dense subset of R and give the first properties. As an example, we show that the Sibson-Thomson element [12] can be obtained by HR^1.

Section 3 is devoted to the proof of sufficient conditions for extending continuously to R a function built on a dense subset. In Section 4 and 5, we present the matrix tools, especially generalized spectral radii, which are used to give a necessary and sufficient condition for convergence. Then in Section 6, we give examples depending on the parameters used in the algorithm. Finally, in Section 7, we produce a few illustrations.

2 Description and first properties of the algorithm HR^1

Our purpose is to define a bivariate function f on a given rectangle $R = I \times J$. We expect that this function will have continuous partial derivatives: $p = f_x$, $q = f_y$. At the beginning of the construction, the only data that are known about these three functions f, p and q are their values at the vertices of R.

Before describing the surface $z = f(x, y)$, we recall the univariate version of this construction, given by Merrien [10].

Suppose that we know the values of a function f and of its first derivative $p = f'$ at the endpoints of an interval I of $\mathbb{R}$. We proceed by induction on n. At step n ($n \geq 0$), $\mathcal{P}_n$ is the regular partition of I in 2^n subintervals of equal lengths $h = |I|/2^n$. If a and b are two consecutive points of $\mathcal{P}_n$, then we compute f and p at the midpoint $\bar{a} = \dfrac{a+b}{2}$ according to the following scheme, which depends on two parameters α ans β:

$$\left.\begin{array}{rcl} f(\bar{a}) & = & \dfrac{f(a)+f(b)}{2} + \alpha h[p(b) - p(a)] \\ p(\bar{a}) & = & (1-\beta)\dfrac{f(b)-f(a)}{h} + \beta \dfrac{p(a)+p(b)}{2} \end{array}\right\} \quad (1)$$

By applying these formulae on ever finer partitions, f and p are defined on a dense set. Moreover there are many values of (α, β) for which f and p are uniformly continuous on I and when this occurs, $p = f'$. Merrien drew attention to two important choices of (α, β), which is the content of next remarks.

Remark 1: If $\alpha = -1/8$, $\beta = -1/2$, then f is the Hermite cubic interpolant.

Remark 2: If $\alpha = -1/8$, $\beta = -1$, then f is the Hermite quadratic spline interpolant with one knot at the midpoint of I.

Let us come back to the rectangle and describe the **algorithm** HR^1.

The values of f, p, q at the vertices of R are specified and the construction depends on 5 parameters $\alpha, \beta, \gamma, \delta, \eta$.

For $n = 0, 1, 2, \ldots$, let us denote by $\mathcal{P}_n$ the regular partition of I in 2^n subintervals and by $\mathcal{Q}_n$ the similar regular partition of J in 2^n subintervals. We proceed by induction on n and we assume that f, p, q are already known on the mesh $\mathcal{P}_n \times \mathcal{Q}_n$. Starting from these values, we define these functions on $\mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}$. Let $h = |I|/2^n$ and $k = |J|/2^n$ and let $(a, c) \in \mathcal{P}_n \times \mathcal{Q}_n$ not on north or east side of the initial rectangle. Then define $b = a + h$, $d = c + k$.

Then $(a, c), (b, c), (b, d)$ and (a, d) are in $\mathcal{P}_n \times \mathcal{Q}_n$. Let $\bar{a} = \frac{a+b}{2}$ and $\bar{c} = \frac{c+d}{2}$. We have to define f, p and q at $(\bar{a}, c), (\bar{a}, d), (a, \bar{c}), (b, \bar{c})$ and $(\bar{a}, \bar{c})$. (see Figure 1).

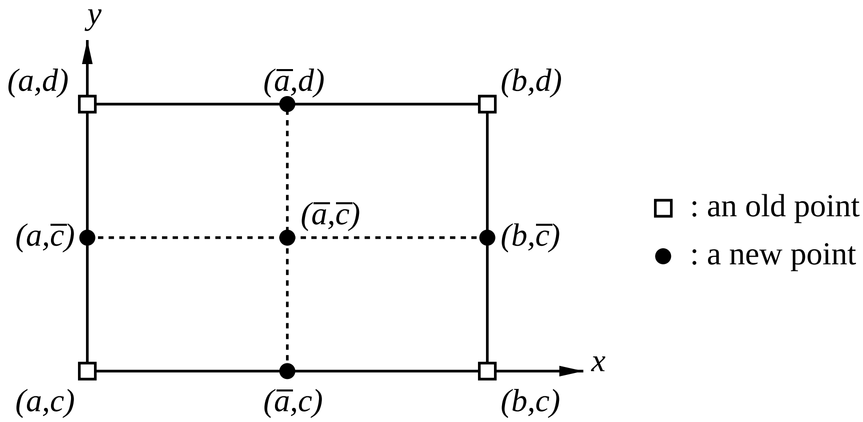

Fig 1: Recursive computation of f.

- At $(\bar{a}, c) \in (\mathcal{P}_{n+1} \setminus \mathcal{P}_n) \times \mathcal{Q}_n$ and similarly on $(\bar{a}, d)$:

$$\left.\begin{aligned} f(\bar{a}, c) &= \frac{f(a,c) + f(b,c)}{2} + \alpha h[p(b,c) - p(a,c)] \\ p(\bar{a}, c) &= (1-\beta)\frac{f(b,c) - f(a,c)}{h} + \beta\frac{p(a,c) + p(b,c)}{2} \\ q(\bar{a}, c) &= \frac{q(a,c) + q(b,c)}{2} \end{aligned}\right\} \tag{2}$$

- At $(a, \bar{c}) \in \mathcal{P}_n \times (\mathcal{Q}_{n+1} \setminus \mathcal{Q}_n)$ and similarly on $(b, \bar{c})$:

$$\left.\begin{aligned} f(a, \bar{c}) &= \frac{f(a,c) + f(a,d)}{2} + \alpha k[q(a,d) - q(a,c)] \\ p(a, \bar{c}) &= \frac{p(a,c) + p(a,d)}{2} \\ q(a, \bar{c}) &= (1-\beta)\frac{f(a,d) - f(a,c)}{k} + \beta\frac{q(a,c) + q(a,d)}{2} \end{aligned}\right\} \tag{3}$$

- At $(\bar{a},\bar{c}) \in (\mathcal{P}_{n+1}\setminus\mathcal{P}_n)\times(\mathcal{Q}_{n+1}\setminus\mathcal{Q}_n)$:

$$\left.\begin{array}{rcl}
f(\bar{a},\bar{c}) & = & \dfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
 & & +\gamma h\dfrac{p(b,c)-p(a,c)+p(b,d)-p(a,d)}{2} \\
 & & +\gamma k\dfrac{q(a,d)-q(a,c)+q(b,d)-q(b,c)}{2} \\
p(\bar{a},\bar{c}) & = & (1-\delta)\dfrac{f(b,c)-f(a,c)+f(b,d)-f(a,d)}{2h} \\
 & & +\delta\dfrac{p(a,c)+p(a,d)+p(b,c)+p(b,d)}{4} \\
 & & +\eta k\dfrac{q(b,d)-q(b,c)+q(a,c)-q(a,d}{h} \\
q(\bar{a},\bar{c}) & = & (1-\delta)\dfrac{f(a,d)-f(a,c)+f(b,d)-f(b,c)}{2k} \\
 & & +\delta\dfrac{q(a,c)+q(a,d)+q(b,c)+q(b,d)}{4} \\
 & & +\eta h\dfrac{p(a,c)-p(a,d)+p(b,d)-p(b,c)}{k}
\end{array}\right\} \quad (4)$$

Remark 3: If we use tensor product to define f,p,q at $(\bar{a},\bar{c})$, then f_{xy} and f_{yx} appear as h tends to 0. We could suppose these derivatives are zero on the vertices of the initial rectangle. This is not compatible with our initial data and the idea of getting a C^1 interpolant.

2.1 An example of Hermite dyadic interpolation

We recall the definition of the quadratic finite element of Sibson-Thomson [12] and we show that it can be constructed by Hermite dyadic interpolation. Let R be a rectangle whose vertices are A,B,C,D, we split R into 4 subrectangles arranged in a St- George pattern:

$$[(A+B)/2,(C+D)/2]\cup[(A+D)/2,(B+C)/2],$$

afterwards we split each subrectangle into a St-Andrew pattern. At the end of the subdivision, R is split into 16 triangular panels (see Figure 2). Sibson and Thomson have shown that for Hermite data (f, f_x, f_y) at the vertices of R, there exists a continuously differentiable function f defined on R, quadratic on each triangular panel, and which interpolates the data. There is only one function which fulfills these conditions, provided it is also assumed that f_y is linear on horizontal edges and f_x is linear on vertical edges.

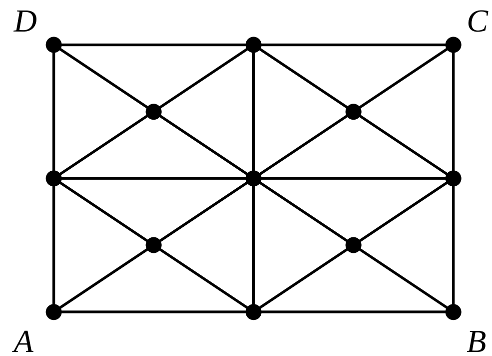

Fig 2: Sibson-Thomson subdivision of a rectangle

Proposition 1 *The Sibson-Thomson solution to the Hermite problem on a rectangle R coincides with the solution given by Hermite dyadic interpolation algorithm HR^1,with parameters $\alpha = \gamma = -1/8$, $\beta = \delta = -1$, $\eta = -1/4$.*

Proof : We assume that $R = [u, u'] \times [v, v']$, and we set $h = u' - u, k = v' - v$. If f is the function defined by Sibson-Thomson, we set $p = f_x$, $q = f_y$. We need to prove that formulae (2-4) are satisfied for $n = 0, 1, \ldots$

Case: $n = 0$,

Formula (2):

On each edge of R, f is a quadratic spline with one knot at the midpoint of the edge. If $A = (u, v), B = (u', v)$, then f restricted to the segment $[A, B]$ is a quadratic spline with one knot at the midpoint $E = (A + B)/2$. This fact and Remark 2 of Merrien in Section 2 show that

$$\left.\begin{array}{rcl} f(E) & = & \frac{f(A)+f(B)}{2} - \frac{h}{8}[p(B)-p(A)] \\ p(E) & = & 2\frac{f(B)-f(A)}{h} - \frac{p(A)+p(B)}{2} \end{array}\right\} \qquad (5)$$

By definition of the Sibson-Thomson element, the value of q to E is specified:

$$q(E) = \frac{q(A)+q(B)}{2} \qquad (6)$$

Since the same argument can be carried for the segment whose endpoints are (u, v') and (u', v'), Formula (2) has been proved for $n = 0$ with $\alpha = -1/8$ and $\beta = -1$.

Formula (3)

On each vertical edge of R, the same arguments can be carried to prove this Formula.

Formula (4)

On the vertical segment $[E, F]$ (where $F = (C + D)/2$), f is a quadratic spline with one knot at the midpoint G of the side. So

$$\left.\begin{array}{rcl} f(G) & = & \frac{f(E)+f(F)}{2} - \frac{k}{8}[q(E)-q(F)] \\ q(G) & = & 2\frac{f(E)-f(F)}{k} - \frac{q(E)+q(F)}{2} \end{array}\right\} \qquad (7)$$

But the values $f(E), f(F), q(E), q(F)$ are already known (Formulae (5-6)). After expansion, it is found that

$$\left.\begin{array}{rcl} f(G) & = & \frac{f(A)+f(B)+f(C)+f(D)}{4} \\ & & -h/16[p(B)-p(A)+p(C)-p(D)] \\ & & -k/16[q(C)+q(D))-q(A)-q(B)] \\ q(G) & = & 2\frac{f(A)+f(B)-f(C)-f(D)}{2k} \\ & & -\frac{h}{4k}[p(B)-p(A)-p(C)+p(D)] \\ & & -\frac{q(A)+q(B)+q(C)+q(D)}{4} \end{array}\right\} \qquad (8)$$

The first and last parts of Formula (4) are proved with $\delta = -1, \eta = -1/4$.

One proceeds similarly on the horizontal segment $[(A+D)/2, (B+C)/2]$ for the evaluation of $p(G)$.

The proof of Formula (2-4) is complete for $n = 0$.

Case: $n > 0$

Let us consider an elementary rectangle R_n coming from the mesh of order $n > 0$. We assume that the vertices of R_n are $A_n = (u_n, v_n), B_n = (u'_n, v_n), C_n = (u'_n, v'_n), D_n = (u_n, v'_n)$. The two diagonals of R_n split R_n in four disjoint triangles. The restriction of f to each of these triangles is a quadratic polynomial; the restrictions of p and of q to the same triangles are linear. This a consequence of the geometry of the Sibson-Thomson subdivision of R. The following properties hold:

- On any side of R_n, f is a quadratic function, p and q are linear.
- The restriction of f to the vertical segment $[(A_n + B_n)/2, (C_n + D_n)/2]$ is a quadratic spline with a unique node at the midpoint of the segment.
- The restriction of f to the horizontal segment $[(A_n + D_n)/2, (C_n + B_n)/2]$ is a quadratic spline with a unique node at the midpoint of the segment.

From these properties, it follows that Formulae (2-4) are true for any n as for $n = 0$. $\diamond$

2.2 Properties of the interpolating process

One of the main properties of the interpolating process is its self-similarity.

Proposition 2 *Let R be a rectangle of the plane, and let $\tilde{R}$ be one of the four smaller rectangles obtained after the bisection of the sides of R. Let us assume that $\{f, p, q\}$ are the three functions that are produced by HR^1 over the rectangle R. Then the three functions $\{\tilde{f}, \tilde{p}, \tilde{q}\}$ that are produced by HR^1 over the rectangle $\tilde{R}$ from the Hermite data: $\tilde{f} = f, \tilde{p} = p, \tilde{q} = q$ at each vertex of $\tilde{R}$, are simply the restrictions of $\{f, p, q\}$ to $\tilde{R}$.*

Another property of the dyadic Hermite interpolation scheme refers to its behavior with respect to a change of scale.

Proposition 3 *Let R be a rectangle, and let us assume that $\{f, p, q\}$ is a triple of functions obtained by HR^1 on R, with parameters $\alpha, \beta, \gamma, \delta, \eta$. Let us consider a change of coordinates: $(x, y) \mapsto T(x, y) = (mx + b, nx + c)$ with $m \neq 0, n \neq 0$. If $\{\tilde{f}, \tilde{p}, \tilde{q}\}$ is the triple of functions obtained by HR^1 on $\tilde{R} = T(R)$, with the same parameters $\alpha, \beta, \gamma, \delta, \eta$, from the Hermite data: $\tilde{f} \circ T = f, \tilde{p} \circ T = p/m, \tilde{q} \circ T = q/n$, at each initial vertex of $\tilde{R}$, then $f = \tilde{f} \circ T, p = m\tilde{p} \circ T, q = n\tilde{q} \circ T$ on R.*

Definition 1. A function g defined on R is said to be *reproduced* by HR^1 on R if the function f coming from the scheme with Hermite data:
$(f, f_x, f_y) = (g, g_x, g_y)$ at each vertex of R coincides with g.

The next question is how to choose the parameters $\alpha, \beta, \gamma, \delta, \eta$ in order to reproduce some specific polynomials.

Proposition 4 *Regardless of the values of $\alpha, \beta, \gamma, \delta, \eta$, any function of the form $a + bx + cy + dxy$ is reproduced by HR^1.*

The main argument of the proof is that the sum of the weights in each formula (2-4) is equal to 1.

Proposition 5

- *x^2 and y^2 are reproduced iff $\alpha = \gamma = -1/8$,*
- *x^2, x^3, y^2, y^3 are reproduced iff $\alpha = \gamma = -1/8,\ \beta = \delta = -1/2$,*
- *x^2, x^2y, xy^2, y^2 are reproduced iff $\alpha = \gamma = -1/8,\ \eta = -1/8$.*

Proof : We consider 6 distinct case.

First case: *x^2 is reproduced by HR^1.*

According to the last proposition and by linearity, the polynomial $P(x, y) =$

$(x-a)(b-x)$ is also reproduced. So if $f = P, p = P_x, q = P_y$, then the first identity in Formula (2) with $n = 0$ is true and this implies that $\alpha = -1/8$. If also the first identity in Formula (4) with $n = 0$ is used, then $\gamma = -1/8$.

Second case: *x^2 and x^3 are reproduced by HR^1.*

The polynomial $P(x, y) = (2x - a - b)^3$ is also reproduced. So if $f = P, p = P_x, q = P_y$, then the second identity in Formula (2) with $n = 0$ is true and this implies that $\beta = -1/2$. If also the second identity in Formula (4) with $n = 0$ is used, then $\delta = -1/2$.

Third case: *x^2 and $x^2 y$ are reproduced by HR^1.*

The polynomial $P(x, y) = (x - a)(b - x)(2y - c - d)$ is also reproduced. So if $f = P, p = P_x, q = P_y$, then the second identity in Formula (4) with $n = 0$ is true and this implies that $\eta = -1/8$.

Fourth case: $\alpha = \gamma = -1/8$.

If $f(x, y) = x^2, p(x, y) = f_x(x, y) = 2x, q = f_y(x, y) = 0$, then Formulae (2-4) can be proved by induction on n. So x^2 is reproduced by HR^1. Of course by symmetry, y^2 is also reproduced.

Fifth case: $\alpha = \gamma = -1/8,\ \beta = \delta = -1/2$.

If $f(x, y) = x^3, p(x, y) = f_x(x, y) = 3x^2, q(x, y) = f_y(x, y) = 0$, then Formulae (2-4) can be proved by induction on n. So x^3 as x^2 is reproduced by HR^1; by symmetry, y^2, y^3 are also reproduced.

Sixth case: $\alpha = \gamma = -1/8,\ \eta = -1/8$.

If $f(x, y) = x^2 y, p(x, y) = f_x(x, y) = 2xy, q = f_y(x, y) = x^2$, then again an induction on n is used for showing that $x^2 y$ is reproduced. ⋄

3 Sufficient conditions for continuous extension

From now on, we assume that $R = [0,1]^2$. It follows that for $n = 0, 1, 2, \ldots$, $\mathcal{P}_n$ is the regular partition of $I = [0,1]$ in 2^n subintervals .

Definition 2. Let $R_n = \mathcal{P}_n \times \mathcal{P}_n$, and let $R_\infty = \bigcup_{n=0}^{\infty} R_n$, the set of *dyadic points* of R. If ϕ is a function which is defined on R_∞, we denote by $\Delta_n(\phi)$ the largest increase of ϕ between two neighbors of the mesh R_n:

$$\Delta_n(\phi) = \max\{|\phi(A) - \phi(B)| : A, B \in R_n, ||A - B|| = 1/2^n\}.$$

Theorem 6 *If ϕ is defined on R_∞ and if $\sum_{n=1}^{\infty} \Delta_n(\phi) < \infty$, then ϕ has a continuous extension on R.*

Proof : For ϕ defined on R_∞, we introduce the sequence ϕ_n of functions defined on R as follows: ϕ_n is the unique function which, on each elementary subsquare of the mesh R_n, is of the form $a + bx + cy + dxy$ and interpolates the values of ϕ at the vertices of the subsquare.

We will show that the following inequality holds:

$$||\phi_{n+1} - \phi_n||_\infty \leq 2\Delta_{n+1}(\phi).$$

Each function $\psi_n = \phi_{n+1} - \phi_n$ is piecewise linear on any horizontal or vertical line. So $||\psi_n||_\infty = \max\{|\psi_n(x,y)| : (x,y) \in R_{n+1}\}$.

Let S be a subsquare whose vertices A, B, C, D belong to R_n, AB is the lower horizontal side of length $1/2^n$, CD is the upper horizontal side of length $1/2^n$.

We set $E = (A + B)/2$,$F = (C + D)/2$,$G = (E + F)/2$. Then

- $\phi_n(E) = [\phi(A) + \phi(B)]/2$ since ϕ_n is linear on any side of S; $\phi_{n+1}(E) = \phi(E)$ by definition; so $|\psi_n(E)| \leq \Delta_{n+1}(\phi)$;

- $|\psi_n(F)| \leq \Delta_{n+1}(\phi)$, similarly;
- $\phi_n(G) = [\phi_n(E)+\phi_n(F)]/2 = [\phi(A)+\phi(B)+\phi(C)+\phi(D)]/4$ since ϕ_n is linear on any vertical segment inside S; $\phi_{n+1}(G) = \phi(G)$, by definition; so $|\psi_n(G)| \leq 2\Delta_{n+1}(\phi)$ since

$$\begin{aligned}\psi_n(G) &= [\phi(G)-\phi(F)]/2 + [\phi(F)-\phi(C)]/4 + [\phi(F)-\phi(D)]/4 \\ &+ [\phi(G)-\phi(E)]/2 + [\phi(E)-\phi(A)]/4 + [\phi(E)-\phi(B)]/4.\end{aligned}$$

It follows that $||\psi_n||_\infty \leq 2\Delta_{n+1}(\phi)$.

By construction, on R_∞, ϕ_n converges pointwise to ϕ. As a telescoping series, $\sum_{n=0}^{N} \psi_n = \phi_{N+1} - \phi_0$. The convergence of $\sum_{n=1}^{\infty} \Delta_n(\phi)$ and Weierstrass criterion (for uniform convergence) show that the sequence ϕ_n converges uniformly to a continuous function on R which coincides with ϕ on R_∞. $\diamond$

Now let us assume that f, p and q are three functions that are defined at the dyadic points of a rectangle R_∞. We are looking for conditions on f ensuring that it has a continuously differentiable extension to R and that $\nabla f = (p, q)$ on R_∞.

We introduce other bounds. Let $h = 1/2^n$ and $E_n(f, p, q)$ be the largest of the following quantities:

- $\Delta_n(p), \Delta_n(q)$,
- the numbers $|\dfrac{f(x+h,y)-f(x,y)}{h} - \dfrac{p(x,y)+p(x+h,y)}{2}|$ where $x, x+h, y \in \mathcal{P}_n$,
- the numbers $|\dfrac{f(x,y+h)-f(x,y)}{h} - \dfrac{q(x,y)+q(x,y+h)}{2}|$ where $x, y, y+h \in \mathcal{P}_n$.

Definition 3. We say that the algorithm HR^1 converges if

1. (f, p, q) built on R_∞ have continuous extensions to R,

2. the extension of f is continuously differentiable,

3. at each dyadic point of R, the functions satisfy $\nabla f = (p, q)$.

Theorem 7 *If f, p, q are defined at the dyadic points of a rectangle R and if $\sum_{n=0}^{\infty} E_n(f, p, q) < \infty$, then the algorithm HR^1 converges.*

Proof : According to Theorem 6, p and q have continuous extensions to R. These continuous extensions are unique, so without loss of generality, we can assume that p and q are defined on R and are continuous.

If $E = \max_n E_n(f, p, q)$, then $\Delta_n(f) \leq (E + \max(||p||_\infty, ||q||_\infty))/2^n$. According to Theorem 6, f has a continuous extension to R. Therefore, we can assume that f is defined and is continuous on R.

By using repeatedly the inequalities

$$|(f(x+h, y) - f(x, y))/h - (p(x+h, y) - p(x, y))/2| \leq E_n(f, p, q),$$

where $x, x+h, y \in \mathcal{P}_n, h = 1/2^n$, it can be shown that

$$(\forall x, x' \in I)(\forall y \in J) \quad f(x', y) - f(x, y) = \int_x^{x'} p(t, y)\, dt,$$

hence $f_x = p$. See Merrien [11] for details. Similarly, we obtain $f_y = q$. $\diamond$

Corollary 8 *If there exists $\kappa \in [0, 1[$ and $c \in \mathbb{R}_+^*$ such that $E_n(f, p, q) \leq c\kappa^n$, then the algorithm HR^1 converges.*

4 Matrix representation of one step of the algorithm HR^1

To study the differences introduced in Section 3, we shall use vectors in $\mathbb{R}^{12}$. For a square obtained at step n, with south-west vertex $(x = i/2^n, y = j/2^n)$

and side length $h = 1/2^n$, we write:

$$U_n(x,y) = \begin{pmatrix} q(x+h,y) - q(x,y) \\ p(x+h,y+h) - p(x+h,y) \\ q(x+h,y+h) - q(x,y+h) \\ p(x,y+h) - p(x,y) \\ p(x+h,y) - p(x,y) \\ q(x+h,y+h) - q(x+h,y) \\ p(x+h,y+h) - p(x,y+h) \\ q(x,y+h) - q(x,y) \\ \frac{f(x+h,y) - f(x,y)}{h} - \frac{p(x+h,y) + p(x,y)}{2} \\ \frac{f(x+h,y+h) - f(x+h,y)}{h} - \frac{q(x+h,y+h) + q(x+h,y)}{2} \\ \frac{f(x+h,y+h) - f(x,y+h)}{h} - \frac{p(x+h,y+h) + p(x,y+h)}{2} \\ \frac{f(x,y+h) - f(x,y)}{h} - \frac{q(x,y+h) + q(x,y)}{2} \end{pmatrix}$$

then we have

Proposition 9

$$\begin{aligned} U_{n+1}(x,y) &= A^{(1)} U_n(x,y), \\ U_{n+1}(x+\frac{h}{2},y) &= A^{(2)} U_n(x,y), \\ U_{n+1}(x+\frac{h}{2},y+\frac{h}{2}) &= A^{(3)} U_n(x,y), \\ U_{n+1}(x,y+\frac{h}{2}) &= A^{(4)} U_n(x,y), \end{aligned}$$

where $A^{(1)}$,$A^{(2)}$,$A^{(3)}$,$A^{(4)}$ *are four matrices in* $\mathbb{R}^{12\text{x}12}$ *depending only on the 5 parameters* $\alpha, \beta, \gamma, \delta, \eta$ *of algorithm* HR^1*.*

Proof : With a computer algebra system, one can verify that

$$A^{(i)} = \begin{pmatrix} A_{11}^{(i)} & A_{12}^{(i)} & A_{13}^{(i)} \\ 0 & A_{22}^{(i)} & A_{23}^{(i)} \\ 0 & A_{32}^{(i)} & A_{33}^{(i)} \end{pmatrix} = \begin{pmatrix} A_{11}^{(i)} & \dots \\ 0 & B^{(i)} \end{pmatrix}$$

with $A_{jk}^{(i)} \in \mathbb{R}^{4\times 4}$ and $B^{(i)} \in \mathbb{R}^{8\times 8}$.

More specifically:

$$A_{11}^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad A_{11}^{(2)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix},$$

$$A_{11}^{(3)} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}, \quad A_{11}^{(4)} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$(A_{12}^{(1)} A_{13}^{(1)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & -\eta & \beta - 1 + \frac{\delta-1}{2} & 0 & \frac{1-\delta}{2} & 0 \\ -\eta & 0 & \eta & 0 & 0 & \frac{1-\delta}{2} & 0 & \beta - 1 + \frac{\delta-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can obtain $(A_{12}^{(i)} A_{13}^{(i)})$ for $i \in \{2,3,4\}$ from $(A_{12}^{(1)} A_{13}^{(1)})$ by permutations of rows and columns.

$$B^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 & 0 & 0 \\ -\eta & \frac{1}{4} & \eta & \frac{1}{4} & 0 & \frac{1-\delta}{2} & 0 & \frac{1-\delta}{2} \\ \frac{1}{4} & \eta & \frac{1}{4} & -\eta & \frac{1-\delta}{2} & 0 & \frac{1-\delta}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta \\ \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ \gamma + \frac{\eta}{2} - 2\alpha & \frac{1}{8}+\gamma & \gamma - \frac{\eta}{2} & \frac{1}{8}+\gamma & 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \\ \frac{1}{8}+\gamma & \gamma - \frac{\eta}{2} & \frac{1}{8}+\gamma & \gamma + \frac{\eta}{2} - 2\alpha & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{pmatrix}$$

$$B^{(2)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 & 0 \\ \frac{1}{4} & -\eta & \frac{1}{4} & \eta & \frac{\delta-1}{2} & 0 & \frac{\delta-1}{2} & 0 \\ -\eta & \frac{1}{4} & \eta & \frac{1}{4} & 0 & \frac{1-\delta}{2} & 0 & \frac{1-\delta}{2} \\ -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ -\frac{1}{8}-\gamma & -\gamma-\frac{\eta}{2}+2\alpha & -\frac{1}{8}-\gamma & -\gamma+\frac{\eta}{2} & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ \gamma+\frac{\eta}{2}-2\alpha & \frac{1}{8}+\gamma & \gamma-\frac{\eta}{2} & \frac{1}{8}+\gamma & 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \end{pmatrix}$$

$$B^{(3)} = \begin{pmatrix} \frac{1}{4} & -\eta & \frac{1}{4} & \eta & \frac{\delta-1}{2} & 0 & \frac{\delta-1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 \\ \eta & \frac{1}{4} & -\eta & \frac{1}{4} & 0 & \frac{\delta-1}{2} & 0 & \frac{\delta-1}{2} \\ -\frac{1}{8}-\gamma & -\gamma-\frac{\eta}{2}+2\alpha & -\frac{1}{8}-\gamma & -\gamma+\frac{\eta}{2} & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ -\gamma+\frac{\eta}{2} & -\frac{1}{8}-\gamma & -\gamma-\frac{\eta}{2}+2\alpha & -\frac{1}{8}-\gamma & 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \end{pmatrix}$$

$$B^{(4)} = \begin{pmatrix} \frac{1}{4} & \eta & \frac{1}{4} & -\eta & \frac{1-\delta}{2} & 0 & \frac{1-\delta}{2} & 0 \\ \eta & \frac{1}{4} & -\eta & \frac{1}{4} & 0 & \frac{\delta-1}{2} & 0 & \frac{\delta-1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 \\ \frac{1}{8}+\gamma & \gamma-\frac{\eta}{2} & \frac{1}{8}+\gamma & \gamma+\frac{\eta}{2}-2\alpha & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ -\gamma+\frac{\eta}{2} & -\frac{1}{8}-\gamma & -\gamma-\frac{\eta}{2}+2\alpha & -\frac{1}{8}-\gamma & 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \\ 0 & 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{pmatrix}$$

$\diamond$

We can immediately deduce:

Corollary 10

$$U_n(x,y) = A^{(d_1)} A^{(d_2)} \dots A^{(d_n)} U_0(0,0),$$

$$\textit{with } d_k \in \{1,2,3,4\}, k = 1 \dots n.$$

The convergence to 0 of (U_n) is proved in the next two sections by studying the products of matrices $A^{(d_1)}A^{(d_2)}\dots A^{(d_n)}$.

5 Convergence and spectral radii of matrix products

For the convergence of the above products of matrices, we shall use the same tools as in Merrien [11]. The problem will be a little more difficult; with the data D at the vertices of the initial square is associated a vector $U_0(0,0)$ but an arbitrary vector $U \in \mathbb{R}^{12}$ does not have necessarily the form $U_0(0,0)$.

We need definitions of spectral radii for products of matrices. Let Σ be a set of matrices of $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. If $\|\ \|$ is a norm on $\mathbb{R}^{\mathrm{n}}$, the norm of a matrix M is $\sup_{\|X\|=1}\|MX\|$. Define:

- $\rho(M)$, the spectral radius of a matrix M,
- $\rho(\Sigma)$, the generalized spectral radius of Σ, $\rho(\Sigma) = \limsup_{k\to+\infty}(\rho_k(\Sigma))^{\frac{1}{k}}$ where
$$\rho_k(\Sigma) = \sup\{\rho(\prod_{i=1}^{k} M_i), M_i \in \Sigma, 1 \le i \le k\},$$
- $\hat\rho(\Sigma)$, the joint spectral radius, $\hat\rho(\Sigma) = \limsup_{k\to+\infty}(\hat\rho_k(\Sigma, \|\ \|))^{\frac{1}{k}}$, where
$$\hat\rho_k(\Sigma, \|\ \|) = \sup\{\|\prod_{i=1}^{k} M_i\|, M_i \in \Sigma, 1 \le i \le k\};$$
remark here that $\hat\rho(\Sigma)$ is independent of the norm used,
- $\nu(\Sigma) = \inf_{\|\ \| \text{ an operator norm}} \sup\{\|A\| : A \in \Sigma\}$.

We shall use the results of Daubechies and Lagarias [2] completed by Berger and Wang [1], then by Elsner [8]:

if Σ is a bounded set then

$$(\rho_k(\Sigma))^{\frac{1}{k}} \leq \rho(\Sigma) = \nu(\Sigma) = \hat{\rho}(\Sigma) \leq (\hat{\rho}_k(\Sigma, \|\ \|))^{\frac{1}{k}}$$

We shall also need a lemma of Berger and Wang :

Lemma 11 *Assume that the matrices $M \in \Sigma$ are all block upper- triangular*

$$M = \begin{pmatrix} M^{(1)} & & * \\ & \ddots & \\ 0 & & M^{(l)} \end{pmatrix},$$

where the $M^{(j)}$ are square matrices. Set $\Sigma^{(j)} = \{M^{(j)} : M \in \Sigma\}$, then:

$$\rho(\Sigma) = \max(\rho(\Sigma^{(1)}), \ldots, \rho(\Sigma^{(l)})).$$

Lemma 12 *Let χ be the linear operator from $I\!R^{12}$ into $I\!R^{12}$ which transforms the 12 data $(f(A), \nabla f(A), \ldots, \nabla f(D))$ at the vertices of the inital square into $U_0(0,0)$. If $\mathcal{V}$ is the subspace generated by $V_1 = (1,0,1,0,0,\ldots,0)^T, V_2 = (0,1,0,1,0,\ldots,0)^T$ and $V_3 = (0,1,1,0,0,\ldots,0)^T$ in $I\!R^{12}$, then:*

1. $\mathcal{V} \oplus Im(\chi) = I\!R^{12}$,
2. *for* $i \in \{1,2,3,4\}, A^{(i)}(Im(\chi)) \subset Im(\chi), A^{(i)}(\mathcal{V}) \subset \mathcal{V}$,
3. *for* $i \in \{1,2,3,4\}, \|A^{(i)}_{|\mathcal{V}}\|_\infty = \frac{1}{2}$.

Proof : If $\chi(D) = 0$ then

$$\begin{cases} p(0,0) = p(1,0) = p(1,1) = p(0,1) = p, \\ q(0,0) = q(1,0) = q(1,1) = q(0,1) = q, \\ f(1,0) = f(0,0) + ph, f(0,1) = f(0,0) + qh, f(1,1) = f(0,0) + ph + qh. \end{cases}$$

This means that the data can be interpolated by a linear polynomial, and $\dim(\mathrm{Ker}(\chi)) = 3$, so that $\dim(\mathrm{Im}(\chi)) = 9$. Precisely $\mathrm{Im}(\chi)$ is the subspace of vectors $U=(x_1,\dots,x_{12})$ such that:

$$\begin{cases} x_1-x_3+x_6-x_8=0 \\ x_2-x_4+x_5-x_7=0 \\ x_9+x_{10}-x_{11}-x_{12}+\frac{1}{2}(x_1+x_3-x_2-x_4)=0 \end{cases}$$

It's easy to see that $V_i \not\subset \mathrm{Im}(\chi)$ and that (V_1,V_2,V_3) is linearly independent. So that

$$\mathrm{Im}(\chi)\oplus\mathcal{V}=\mathbb{R}^{12}.$$

Clearly if $U_1=A^{(i)}U_0(0,0)$ with $i\in\{1,2,3,4\}$ then $U_1\in\mathrm{Im}(\chi)$, so that

$$A^{(i)}(\mathrm{Im}(\chi))\subset\mathrm{Im}(\chi).$$

Now for $i\in\{1,2,3,4\}$, $A^{(i)}V_1=\frac{1}{2}V_1, A^{(i)}V_2=\frac{1}{2}V_2$ and
$A^{(1)}V_3=\frac{1}{4}V_3, A^{(2)}V_3=\frac{1}{4}(V_2+V_3), A^{(3)}V_3=\frac{1}{4}(V_1+V_2+V_3)$,
$A^{(4)}V_3=\frac{1}{4}(V_1+V_3)$, then :

$$A^{(i)}(\mathcal{V})\subset\mathcal{V}.$$

If $V=x_1V_1+x_2V_2+x_3V_3=(x_1,x_2+x_3,x_1+x_3,x_2)^T\in\mathcal{V}$ with $\|V\|_\infty=1$ then $A^{(1)}V=(\dfrac{x_1}{2},\dfrac{x_2}{2}+\dfrac{x_3}{4},\dfrac{x_1}{2}+\dfrac{x_3}{4},\dfrac{x_2}{2})^T$, so that

$$\|A^{(1)}V\|_\infty\le\max(\frac{|x_1|}{2},\frac{|x_2|}{4}+\frac{|x_2+x_3|}{4},\frac{|x_1|}{4}+\frac{|x_1+x_3|}{4},\frac{|x_2|}{2})\le\frac{1}{2},$$

with equality for $V=V_1$. This gives $\|A^{(1)}_{|\mathcal{V}}\|_\infty=\frac{1}{2}$. The other norms are evaluated similarly. ⋄

To use the spectral radii, we choose $\Sigma=\{A^{(1)},A^{(2)},A^{(3)},A^{(4)}\}$ where the matrices are defined in Proposition 9.

Proposition 13 *The algorithm HR^1 is convergent if and only if $\rho(\Sigma)<1$*

Proof : Since $\rho(\Sigma) = \nu(\Sigma) < 1$, there exists an operator norm $\| \ \|$ for which $\kappa = \max(\|A^{(i)}\|, i = 1, 2, 3, 4) < 1$. So that for any vector

$$U_n(x, y) = A^{(d_1)} A^{(d_2)} \dots A^{(d_n)} U_0(0, 0),$$

we have $\|U_n(x, y)\| \leq c\kappa^n$ and since the norms are equivalent: $\|U_n(x, y)\|_\infty \leq c'\kappa^n$. According to Theorem 7 and its Corollary the algorithm HR^1 converges.

Conversely, if the algorithm converges, we know that p, q and f are continuous on the square with $f_x = p$ and $f_y = q$; morover p and q are uniformly continuous. For any data D and the corresponding vector $U_0(0, 0)$, it's easy to prove, by using uniform continuity and Taylor expansions, that $U_n(x, y)$ tends to 0 as n tends to $+\infty$, which can be resumed in: $\|U_n(x, y)\|_\infty \leq \varepsilon(n, U_0)$ with $\lim_{n \to +\infty} \varepsilon(n, U_0) = 0$.

Let us choose a basis $\mathcal{B} = (V_1, V_2, V_3, \dots, V_{12})$, composed of vectors V, adapted to the decomposition $\mathcal{V} \oplus Im(\chi) = \mathbb{R}^{12}$, where (V_1, V_2, V_3) are defined in the preceding proposition, then:

$$\|A^{(d_1)} A^{(d_2)} \dots A^{(d_n)} V_i\|_\infty \leq \frac{1}{2^n}, \text{ for } i \in \{1, 2, 3\}.$$

And for any vector V of the basis $\mathcal{B}$, we have proved that:

$$\|A^{(d_1)} A^{(d_2)} \dots A^{(d_n)} V\|_\infty \leq \varepsilon(n, V).$$

Now if $U \in \mathbb{R}^{12}$ with $\|U\|_\infty = 1$, $U = \sum_{i=1}^{12} \lambda_i V_i$, then $\max |\lambda_i|$ is bounded independently of U. So that we have:

$$\|A^{(d_1)} A^{(d_2)} \dots A^{(d_n)} U\|_\infty \leq \sum_{i=1}^{12} |\lambda_i| \varepsilon(n, V_i) \leq \varepsilon(n), \text{ with } \lim_{n \to +\infty} \varepsilon(n) = 0.$$

and

$$\|A^{(d_1)} A^{(d_2)} \dots A^{(d_n)}\|_\infty \leq \varepsilon(n).$$

There exists an integer k such that $\varepsilon(k) < 1$, therefore $\hat{\rho}_k(\Sigma, \|\ \|_\infty) < 1$. As $\rho(\Sigma) \leq (\hat{\rho}_k(\Sigma, \|\ \|_\infty))^{\frac{1}{k}}$, we get the result. $\diamond$

Set $\sigma = \{B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}\}$.

Corollary 14 *The algorithm HR^1 is convergent if and only if $\rho(\sigma) < 1$.*

Proof : We know that $A^{(i)} = \begin{pmatrix} A_{11}^{(i)} & * \\ 0 & B^(i) \end{pmatrix}$. Using Lemma 11 we obtain:

$$\rho(\Sigma) = \max(\rho(A_{11}^{(1)}, A_{11}^{(2)}, A_{11}^{(3)}, A_{11}^{(4)}), \rho(\sigma)).$$

A direct computation gives $\|A_{11}^{(i)}\|_\infty = \frac{1}{2}$, so that $\rho(A_{11}^{(1)}, A_{11}^{(2)}, A_{11}^{(3)}, A_{11}^{(4)}) \leq \frac{1}{2}$. Now $\rho(\Sigma) < 1$ if and only if $\rho(\sigma) < 1$. With the preceding proposition, we get the result. $\diamond$

6 Necessary and/or sufficient conditions of convergence

This last necessary and sufficient condition is important, but the computation of the spectral radii is difficult. Gripenberg [9] gives algorithms to find an arbitrary small interval that contains the joint spectral radius of a finite set of matrices, but we whould like to find conditions of convergence depending on the parameters. Using again the inequalities:

$$(\rho_k(\sigma))^{\frac{1}{k}} \leq \rho(\sigma) = \nu(\sigma) = \hat{\rho}(\sigma) \leq (\hat{\rho}_k(\sigma, \|\ \|))^{\frac{1}{k}}$$

we immediately get:
if there exists k such that $\rho_k(\sigma)) > 1$ then the algorithm HR^1 diverges;
if there exists an operator norm $\|\ \|$ and k such that $\hat{\rho}_k(\sigma, \|\ \|) < 1$ then the algorithm converges.

We shall study $\rho(\sigma)$ where $\sigma = \{B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}\}$ with $B^{(i)} = \begin{pmatrix} A_{22}^{(i)} & A_{23}^{(i)} \\ A_{32}^{(i)} & A_{33}^{(i)} \end{pmatrix}$.
Set $\sigma_j = \{A_{jj}^{(1)}, A_{jj}^{(2)}, A_{jj}^{(3)}, A_{jj}^{(4)}\}$ for $j = 2$ or $j = 3$.

Proposition 15 *If $\alpha = \gamma = -1/8, \eta = -1/4$,*

1. *the algorithm HR^1 converges if and only if $\rho(\sigma_3) < 1$;*
2. *for $\beta = \delta$, the algorithm HR^1 converges if and only if $-3 < \beta < 1$;*
3. *for $\beta \neq \delta$,*
   if $-3 < \beta < 1$ and $-3 < \delta < 1$, the algorithm HR^1 converges ;
   if $\beta \notin]-3, 1[$ or $\delta \notin]-5, 3[$, the algorithm HR^1 is not convergent ie $f \notin C^1$.

Proof : With the above conditions on the parameters, we obtain:

$$B^{(i)} = \begin{pmatrix} A_{22}^{(i)} & * \\ 0 & A_{33}^{(i)} \end{pmatrix}$$

where:

$$A_{22}^{(1)} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_{22}^{(2)} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

$$A_{22}^{(3)} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \quad A_{22}^{(4)} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$A_{33}^{(1)} = \begin{pmatrix} \frac{1+\beta}{2} & 0 & 0 & 0 \\ 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \\ \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & 0 & 0 & \frac{1+\beta}{2} \end{pmatrix}, \quad A_{33}^{(2)} = \begin{pmatrix} \frac{1+\beta}{2} & 0 & 0 & 0 \\ 0 & \frac{1+\beta}{2} & 0 & 0 \\ \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & \frac{1+\delta}{4} & 0 & \frac{1+\beta}{2} \end{pmatrix},$$

$$A_{33}^{(3)} = \begin{pmatrix} \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & \frac{1+\beta}{2} & 0 & 0 \\ 0 & 0 & \frac{1+\beta}{2} & 0 \\ 0 & \frac{1+\delta}{4} & 0 & \frac{1+\beta}{2} \end{pmatrix}, \quad A_{33}^{(4)} = \begin{pmatrix} \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} & 0 \\ 0 & \frac{1+\delta}{4} & 0 & \frac{1+\delta}{4} \\ 0 & 0 & \frac{1+\beta}{2} & 0 \\ 0 & 0 & 0 & \frac{1+\beta}{2} \end{pmatrix}.$$

Now, for all $i \in \{1,2,3,4\}$, $\|A_{22}^{(i)}\|_2 = \dfrac{\sqrt{2}}{2}$,
$\rho(A_{33}^{(i)}) = \max(|\dfrac{1+\beta}{2}|, |\dfrac{1+\delta}{4}|)$ and $\|A_{33}^{(i)}\|_\infty = \max(|\dfrac{1+\beta}{2}|, |\dfrac{1+\delta}{2}|)$.
Using the inequalities on spectral radii, we have $\rho(\sigma_2) \leq \dfrac{\sqrt{2}}{2}$ and

$$\max(|\frac{1+\beta}{2}|, |\frac{1+\delta}{4}|) \leq \rho(\sigma_3) \leq \max(|\frac{1+\beta}{2}|, |\frac{1+\delta}{2}|). (*)$$

From lemma 11, we know that $\rho(\sigma) = \max(\rho(\sigma_2), \rho(\sigma_3))$, so that $\rho(\sigma) < 1$ if and only if $\rho(\sigma_3) < 1$.

If $\beta = \delta$ then from inequalities $(*)$ above, $\rho(\sigma_3) = |\dfrac{1+\beta}{2}|$ and $\rho(\sigma) < 1$ if and only if $|\dfrac{1+\beta}{2}| < 1$, which gives the second result.

The third result with $\beta \neq \delta$ is a direct consequence of the inequalities $(*)$. $\diamond$

To compute $\rho(\sigma_3)$, note that if we suppose that the $A_{33}^{(i)}$ are associated with the operators written in the canonical basis of $\mathbb{R}^4$,$\{e_1, e_2, e_3, e_4\}$ then, if we write these operators in $\{e_1, e_3, e_2, e_4\}$, it is easy to see that the matrices are all block diagonal form and using again Lemma 11

$$\rho(\sigma_3) = \rho(\begin{pmatrix} \frac{1+\beta}{2} & 0 \\ \frac{1+\delta}{4} & \frac{1+\delta}{4} \end{pmatrix}, \begin{pmatrix} \frac{1+\delta}{4} & \frac{1+\delta}{4} \\ 0 & \frac{1+\beta}{2} \end{pmatrix}).$$

Example 1: For the Sibson-Thomson element, $\alpha = \gamma = -1/8, \beta = \delta = -1, \eta = -1/4$, then $A_{32}^{(i)} = 0, A_{33}^{(i)} = 0$ and $\rho(\sigma) \leq \frac{\sqrt{2}}{2}$. As already proved the algorithm HR^1 converges.

Proposition 16 *If there exists an operator norm $\| \ \|$ on $\mathbb{R}^4$ such that*
$\forall i \in \{1,2,3,4\}$*:*
$\|A_{22}^{(i)}\| < 1, \|A_{33}^{(i)}\| < 1,$ *and* $\|A_{23}^{(i)}\|.\|A_{32}^{(i)}\| < (1 - \|A_{22}^{(i)}\|)(1 - \|A_{33}^{(i)}\|)$
then the algorithm HR^1 converges.

Proof : For $V \in \mathbb{R}^8, \mathrm{V} = \begin{pmatrix} X \\ Y \end{pmatrix}$ with $X, Y \in \mathbb{R}^4$,
define $\|V\|' = \|X\| + \lambda\|Y\|$ where $\lambda \in \mathbb{R}_+^*$ is to be chosen. Then

$$\begin{aligned} \|B^{(i)}V\|' &= \|A_{22}^{(i)}X + A_{23}^{(i)}Y\| + \lambda\|A_{32}^{(i)}X + A_{33}^{(i)}Y\| \\ &\leq (\|A_{22}^{(i)}\| + \lambda\|A_{32}^{(i)}\|)\|X\| + (\frac{\|A_{23}^{(i)}\|}{\lambda} + \|A_{33}^{(i)}\|)\lambda\|Y\| \\ &\leq \max(\|A_{22}^{(i)}\| + \lambda\|A_{32}^{(i)}\|, \frac{\|A_{23}^{(i)}\|}{\lambda} + \|A_{33}^{(i)}\|)\|V\|' \end{aligned}$$

So that $\|B^{(i)}\|' < 1$ as soon as
$\|A_{22}^{(i)}\| + \lambda\|A_{32}^{(i)}\| < 1$ and $\frac{\|A_{23}^{(i)}\|}{\lambda} + \|A_{33}^{(i)}\| < 1$, which may be written:

$$\lambda\|A_{32}^{(i)}\| < 1 - \|A_{22}^{(i)}\|, \frac{\|A_{23}^{(i)}\|}{\lambda} < 1 - \|A_{33}^{(i)}\|.$$

Now using the hypothesis of the Theorem, we are able to choose λ such that these inequalities hold. $\diamond$

Example 2: We shall use the norm $\| \ \|_\infty$ on $\mathbb{R}^4$.
Let $\alpha = -1/16, \gamma = -1/8, \beta = \delta = -1, \eta = 0$ then:

$$\|A_{22}^{(i)}\|_\infty = 1/2, \|A_{23}^{(i)}\|_\infty = 2, \|A_{32}^{(i)}\|_\infty = 1/8, \|A_{33}^{(i)}\|_\infty = 0.$$

So that $(1 - \|A_{22}^{(i)}\|_\infty)(1 - \|A_{33}^{(i)}\|_\infty) - \|A_{23}^{(i)}\|_\infty.\|A_{32}^{(i)}\|_\infty = 0.25 > 0$. The algorithm HR^1 converges. (see Figure 5).

Example 3: We shall suppose $\alpha = \gamma = -1/8$ and $\beta = \delta$ so that the algorithm depends on two parameters β and η. We shall use the norm $\| \ \|_2$ on $\mathbb{R}^4$. Then a direct computation gives:

$\|A_{22}^{(i)}\|_2^2$

$$= \max\left(\frac{3}{16}+\eta^2+\frac{\sqrt{5+32\eta^2+256\eta^4+32\eta}}{16}, \frac{3}{16}+\eta^2+\frac{\sqrt{5+32\eta^2+256\eta^4-32\eta}}{16}\right)$$

$$\|A_{23}^{(i)}\|_2 = |1-\beta|\frac{\sqrt{10}+\sqrt{2}}{4}, \|A_{32}^{(i)}\|_2 = |\frac{1}{8}+\frac{\eta}{2}|\sqrt{2}, \|A_{33}^{(i)}\|_2 = |1+\beta|\frac{\sqrt{10}+\sqrt{2}}{8}.$$

The condition $\|A_{22}^{(i)}\|_2 < 1$ gives $\frac{1}{28} - \frac{3\sqrt{2}}{7} < \eta < -\frac{1}{28} + \frac{3\sqrt{2}}{7}$ with $-\frac{1}{28} + \frac{3\sqrt{2}}{7} \simeq 0.57$.

To get $\|A_{33}^{(i)}\|_2 < 1$, we need $-1-\sqrt{10}+\sqrt{2} < \beta < -1+\sqrt{10}-\sqrt{2}$ with $\sqrt{10}-\sqrt{2} \simeq 1.74$.

For $\beta \in [-1.5, 0.2]$ and $\eta \in [-0.3, -0.12]$, we have drawn the surface

$$s(\beta,\eta) = (1-\|A_{22}^{(i)}\|_2).(1-\|A_{33}^{(i)}\|_2) - \|A_{32}^{(i)}\|_2.\|A_{23}^{(i)}\|_2.$$

The algorithm HR^1 converges if $s(\beta,\eta) > 0$. (see Figure 3 for $s(\beta,\eta)$ and Figure 6 for the surfaces)

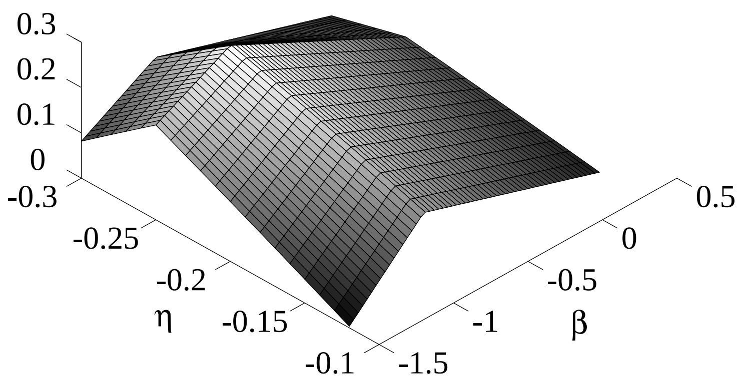

Fig 3: Graph of $s(\beta,\eta)$.

7 Examples

On the square $[0,1]^2$, we have interpolated the data:

$$\begin{array}{llll} f(0,0)=0, & f(1,0)=-0.2, & f(0,1)=0.3, & f(1,1)=1, \\ f_x(0,0)=-2, & f_x(1,0)=-1, & f_x(0,1)=0, & f_x(1,1)=1, \\ f_y(0,0)=0, & f_y(1,0)=1.2, & f_y(0,1)=0, & f_y(1,1)=2/3. \end{array}$$

by the algorithm HR^1. We stopped the process for $n=5$. So that the values of f, f_x, f_y are evaluated at $(2^5+1)\times(2^5+1)$ points of $[0,1]^2$. Then we have drawn the surfaces f, p, q and the level curves of f.

7.1 The Sibson-Thomson element

We choose $\alpha=\gamma=-1/8, \beta=\delta=-1, \eta=-1/4$. f_x and f_y are linear on each subtriangle.

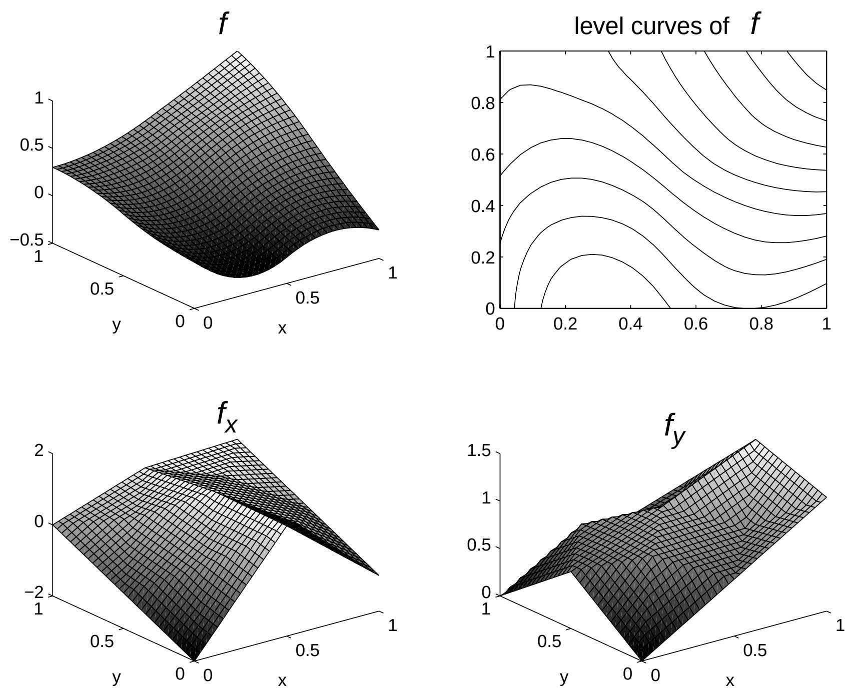

Fig 4: Graphs of f, f_x, f_y and level curves of f.

7.2 $\alpha = -1/16, \gamma = -1/8, \beta = \delta = -1, \eta = 0$

This is an illustration of Example 2. The functions f_x and f_y are continuous on $[0,1]^2$ but irregular. On the sides $x = 0$ and $x = 1$, the function f_x is linear and piecewise linear on $x = 1/2$ ... and similarly for f_y on $y = 0$ ...

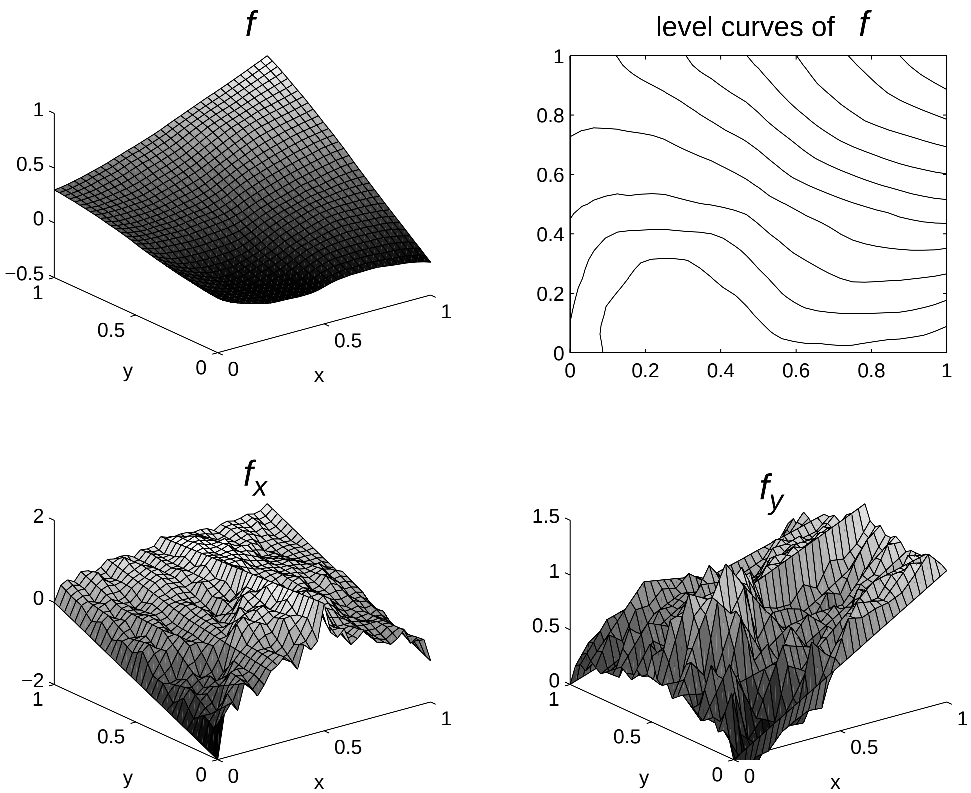

Fig 5: Graphs of f, f_x, f_y and level curves of f with $\eta = 0$.

7.3 $\alpha = \gamma = -1/8, \beta = \delta = -0.6, \eta = -0.15$

This is an illustration of Example 3. $\eta = -0.15$ and the functions f_x, f_y are less irregular than in the preceding example.

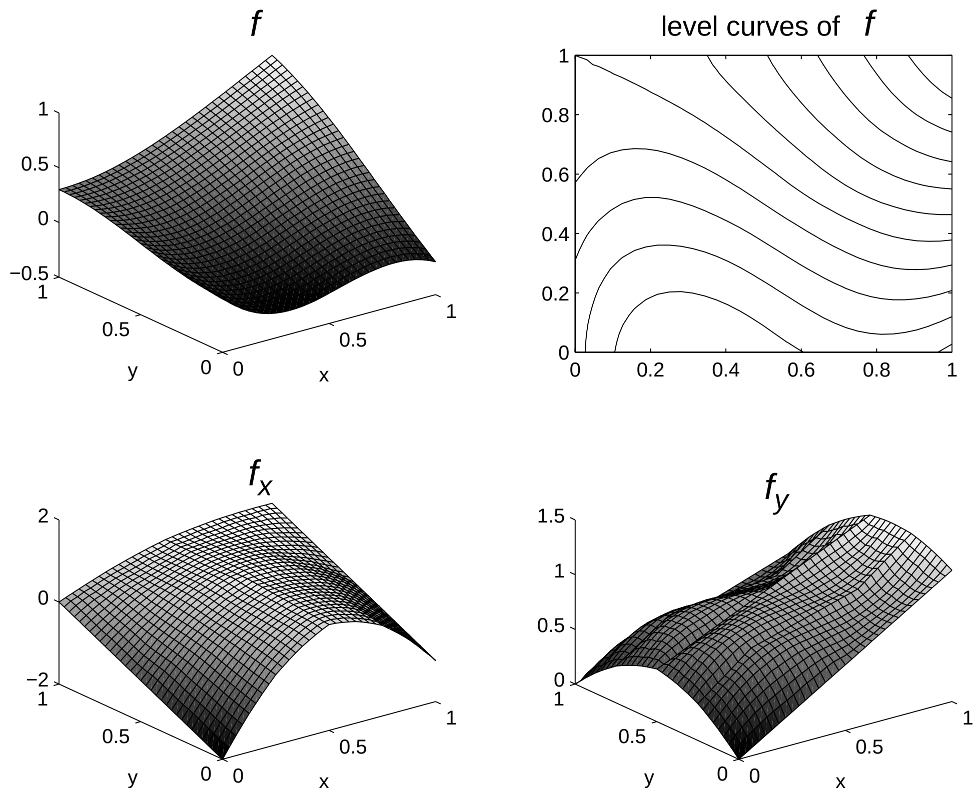

Fig 6: Graphs of f, f_x, f_y and level curves of f

8 Conclusion

We considered dyadic Hermite interpolations on a rectangular mesh under the assumption that the process of interpolation is invariant under any symmetry of the original mesh. The simplest Hermite type subdivision schemes involve at most 5 parameters $(\alpha, \beta, \gamma, \delta, \eta)$ and it is possible to check for which values of these parameters, one gets an interpolating C^1 function for arbitrary Hermite data.

From a set of Hermite data, we got parametric surfaces f, f_x and f_y where $x \in [0, 1]$ and $y \in [0, 1]$. In contrast with tensor products, no second order mixed partial derivatives are used. We conclude by saying that other Hermite subdivision schemes can be considered. It is an open question to know if our techniques can be extended to this situation.

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