A 4-Point Hermite Subdivision Scheme

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Abstract. A subdivision scheme based on 4 points with Hermite data (function and first derivatives) on $\mathbb{Z}$ is studied. For a large region in the parameter space, the scheme is C^1 convergent or at least is convergent in the space of Schwartz distributions. The Fourier transform of any interpolating function can be computed through products of matrices of order 2. The main tools for proving these results are the Paley-Wiener-Schwartz theorem on the characterization of the Fourier transforms of distributions with compact support, and a theorem of Heil-Colella about the convergence of some products of matrices.

§1. Introduction

Hermite interpolatory subdivision schemes have been introduced by Merrien [7]. He, Dyn and Levin [3,4] studied the convergence of these schemes to regular functions. In this paper, we would like to consider the most general case of a 4-point Hermite interpolatory symmetrical scheme using function and first derivatives values. Such a scheme will be called HS41. We will study the conditions giving a C^1 interpolant (Section 2), and weaker conditions allowing convergence in the space of Schwartz distributions. The last task will be done by a computation of Fourier transforms (Section 3). This harmonic analysis provides an additional tool to study the scheme, and allows extension to functions which are not necessarily of class C^1. To get the convergence in distributions, we will use a result proved by Heil and Colella [6] on the convergence of some infinite products of matrices arising in matrix refinement equations.

§2. The Hermite Subdivision Scheme HS41

We assume that the function f and its first derivative p are known on $\mathbb{Z}$. Precisely, we have two sequences $\{y_k, y'_k\}_{k\in\mathbb{Z}}$, and we suppose $f(k) = y_k$ and $p(k) = y'_k$. We build f and p on $D_n = \mathbb{Z}/2^n$ by induction. At step n, if

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$\frac{i-1}{2^n}, \frac{i}{2^n}, \frac{i+1}{2^n}, \frac{i+2}{2^n}$ are four successive points of D_n, we compute f and p at $x = \frac{2i+1}{2^{n+1}}$ by the formulae:

$$\begin{aligned} f(\frac{2i+1}{2^{n+1}}) &= a_1[f(\frac{i+1}{2^n}) + f(\frac{i}{2^n})] + b_1[f(\frac{i+2}{2^n}) + f(\frac{i-1}{2^n})] \\ &\quad + \frac{c_1}{2^n}[p(\frac{i+1}{2^n}) - p(\frac{i}{2^n})] + \frac{d_1}{2^n}[p(\frac{i+2}{2^n}) - p(\frac{i-1}{2^n})], \\ p(\frac{2i+1}{2^{n+1}}) &= 2^n a_2[f(\frac{i+1}{2^n}) - f(\frac{i}{2^n})] + 2^n b_2[f(\frac{i+2}{2^n}) - f(\frac{i-1}{2^n})] \\ &\quad + c_2[p(\frac{i+1}{2^n}) + p(\frac{i}{2^n})] + d_2[p(\frac{i+2}{2^n}) + p(\frac{i-1}{2^n})]. \end{aligned} \tag{1}$$

Hence f and p are defined on D_{n+1}. The construction depends on eight parameters. When we reiterate the process, we define f and p on the set of dyadic numbers $D_\infty = \bigcup D_n$ which is dense in $\mathbb{R}$.

For $b_1 = d_1 = b_2 = d_2 = 0$, we recover 2-point subdivision schemes. We call them HS21. They have been studied by Merrien [7] to get a C^1 function f with $f' = p$. Recently, Dubuc and Merrien [2] gave a new study of the convergence of these schemes in the space of Schwartz distributions. We want a generalization of these results for new 4-point schemes.

Now, suppose that f and p built on D_∞ can be extended to $\mathbb{R}$ in smooth enough functions with $f' = p$. As n tends to ∞, a Taylor expansion of $f(\frac{1}{2^n})$ and $p(\frac{1}{2^n})$ at the origin gives necessary conditions on the parameters. This is to be connected to the reproducibility of polynomials. Like Dyn and Levin in [3], we will say that the scheme is C^r if, for any data $\{y_k, y'_k\}_{k\in\mathbb{Z}}$, there exist two functions, $f \in C^r(\mathbb{R})$ and $p \in C^{r-1}(\mathbb{R})$ such that $f' = p$ and $f(k) = y_k, p(k) = y'_k$. Dyn and Levin have proved that if the scheme is C^r, then it reproduces polynomials of degree less or equal to r. More precisely, if $\{y_k, y'_k\}_{k\in\mathbb{Z}}$ are two sequences such that there exists a polynomial P of degree less than or equal to r with $P(k) = y_k, P'(k) = y'_k$, then $f = P$ and $p = P'$ on D_∞.

Proposition 1. *Assume that the scheme is C^r. Then the following conditions are necessary:*

for $r = 0$, $a_1 + b_1 = 1/2$, for $r = 1$, $a_2 + 3b_2 + 2c_2 + 2d_2 = 1$,
for $r = 2$, $2a_1 - c_1 - 3d_1 = 9/8$, for $r = 3$, $6b_2 + c_2 + 13d_2 = -1/4$,
for $r = 4$, $-c_1 + 3d_1 = 9/64$, for $r = 5$, $-c_2 + 9d_2 = 9/32$,
for $r = 6$, $a_1 = 243/512, b_1 = 13/512, c_1 = -81/512, d_1 = -3/512$,
for $r = 7$, $a_2 = 405/256, b_2 = 5/256, c_2 = -81/256, d_2 = -1/256$.

2.1. Convergence and Smoothness Analyses of HS41

We now study when the scheme is C^1. The necessary conditions given above imply that $a_1 + b_1 = 1/2$ and $a_2 + 3b_2 + 2c_2 + 2d_2 = 1$. Set

$$U_n^i = \begin{pmatrix} p(\frac{i+1}{2^n}) - p(\frac{i}{2^n}) \\ 2^n[f(\frac{i+1}{2^n}) - f(\frac{i}{2^n})] - \frac{p(\frac{i+1}{2^n})+p(\frac{i}{2^n})}{2} \end{pmatrix}.$$

With the help of a computer algebra system, we immediately get

Proposition 2.

$$U_{n+1}^{2i} = A_1 U_n^i + B_1 U_n^{i+1} + C_1 U_n^{i-1},$$
$$U_{n+1}^{2i+1} = A_{-1} U_n^i + B_{-1} U_n^{i+1} + C_{-1} U_n^{i-1},$$

with

$$A_j = \begin{pmatrix} \frac{1}{2} & j(a_2+b_2) \\ j(-2a_1+\frac{5}{4}+2c_1+2d_1) & 1-\frac{a_2+b_2}{2} \end{pmatrix},$$
$$B_j = \begin{pmatrix} j(\frac{b_2}{2}+d_2) & jb_2 \\ j(-a_1+\frac{1}{2}+2d_1)-\frac{b_2}{4}-\frac{d_2}{2} & j(-2a_1+1)-\frac{b_2}{2} \end{pmatrix}, \qquad j=\pm 1,$$
$$C_j = \begin{pmatrix} -j(\frac{b_2}{2}+d_2) & jb_2 \\ j(-a_1+\frac{1}{2}+2d_1)+\frac{b_2}{4}+\frac{d_2}{2} & j(2a_1-1)-\frac{b_2}{2} \end{pmatrix}.$$

Theorem 3. *If there exists a matrix norm* $\|\cdot\|$ *on* $\mathbb{R}^{2\times 2}$ *such that* $\|A_j\|+\|B_j\|+\|C_j\|<1, j=\pm 1$, *then the scheme is* C^1.

Proof: The functions p and f are built on D_∞. We want to extend them to $\mathbb{R}$ and prove that $f' = p$. We will do it on $[0,1]$, and the extension to $\mathbb{R}$ will be obvious.

Set $\kappa = \max(\|A_j\|+\|B_j\|+\|C_j\|, j=\pm 1)$. Then for all $n \in \mathbb{N}$ and for all $i \in \{0,\dots,2^n-1\}$, $\|U_n^i\| \le \gamma_1 \kappa^n$, where the constant γ_1 depends on the initial data on $[-3,3]\cap\mathbb{Z}$. Let γ_2 be a real number such that for any vector $V \in \mathbb{R}^2, \|V\|_\infty \le \gamma_2 \|V\|$.

Firstly, for $n \in \mathbb{N}$, let p_n be the continuous piecewise linear function on $[0,1]$ defined by $p_n(i2^{-n}) = p(i2^{-n})$, $i \in \{0,\dots,2^n\}$. Then we have

$$\begin{aligned}\|p_{n+1}-p_n\|_\infty &= \sup_{i\in\mathbb{Z}} |p((2i+1)2^{-n-1}) - \frac{1}{2}(p((i+1)2^{-n})+p(i2^{-n}))| \\ &\le \frac{1}{2}(\|U_{n+1}^{2i+1}\|_\infty + \|U_{n+1}^{2i}\|_\infty) \\ &\le \frac{1}{2}\gamma_2(\|U_{n+1}^{2i+1}\| + \|U_{n+1}^{2i}\|) \le \gamma_1\gamma_2\kappa^{n+1}.\end{aligned}$$

Hence we can deduce that p_n is a Cauchy sequence in $C([0,1])$. Therefore it has a continuous limit that we still call p since it is an extension.

Secondly, let us set $\varphi(x) = f(0) + \int_0^x p(t)dt$. φ is in $C^1(\mathbb{R})$ with $\varphi' = p$. Let us prove that $\varphi = f$.

Given $x \in D_\infty \cap [0,1]$ and $\epsilon > 0$, there exists n such that $x \in D_n$; since $x \in D_{n'}$, for all $n' \ge n$, we can choose n as large as we need. Since p is uniformly continuous on $[0,1]$, for n large enough and $i \in \{0,\dots,2^n-1\}$, we have

$$\forall t \in [i2^{-n},(i+1)2^{-n}], \quad |p(t) - (p(i2^{-n}) + p((i+1)2^{-n}))/2| \le \epsilon.$$

Similarly we suppose n large enough to ensure $\gamma_1\gamma_2\kappa^n \leq \epsilon$.
Writing $x = k2^{-n}, k \in \{0, \ldots, 2^n\}$, we have

$$\begin{aligned}\varphi(x) - f(x) &= f(0) + \int_0^x p(t)dt - f(x)\\ &= \sum_{i=0}^{k-1}\{\int_{i2^{-n}}^{(i+1)2^{-n}} p(t)dt - [f((i+1)2^{-n}) - f(i2^{-n})]\}\\ &= \sum_{i=0}^{k-1}\int_{i2^{-n}}^{(i+1)2^{-n}} [p(t) - \frac{p((i+1)2^{-n}) + p(i2^{-n})}{2}]dt\\ &+ \sum_{i=0}^{k-1}\{\frac{p((i+1)2^{-n}) + p(i2^{-n})}{2^{n+1}} - [f((i+1)2^{-n}) - f(i2^{-n})]\}.\end{aligned}$$

For all $i \in \{0, \ldots, 2^n\}$, $\|U_n^i\|_\infty \leq \gamma_2\|U_n^i\| \leq \gamma_1\gamma_2\kappa^n$, and with the hypotheses on n, we can deduce that $| \varphi(x) - f(x) | \leq \sum_{i=0}^{k-1} 2^{-n}\epsilon + 2^{-n}\sum_{i=0}^{k-1}\epsilon \leq 2\epsilon$. As ϵ can be chosen arbitrarily small, we obtain $\varphi = f$ on $D_\infty \cap [0,1]$. Therefore φ is a continuous extension of f on $[0,1]$, $f \in C^1([0,1])$ and $f' = p$.
□

Theorem 3 gives a sufficient condition for the operator $U_n \to U_{n+1}$ to be contractive. A weaker condition for the C^1-convergence of the scheme can be obtained when the operator $U_n \to U_{n+m}$ is a contraction for an integer $m > 1$.

2.2. Examples

Before giving examples, we introduce a norm on $\mathbb{R}^2$ by $\|X\|_\theta = |x_1| + \theta|x_2|$ for $X = (x_1, x_2)^T$ where θ is a real positive number. Then it is easy to prove that for $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$, $\|M\|_\theta = \max_{\|X\|_\theta = 1}(\|MX\|_\theta) = \max(|m_{11}| + \theta|m_{21}|, \frac{|m_{12}|}{\theta} + |m_{22}|)$. In some cases, it is convenient to know that we can find a $\theta > 0$ to get $\|M\|_\theta < 1$ if and only if $|m_{11}| < 1, |m_{22}| < 1$ and $|m_{12}| \cdot |m_{21}| < (1 - |m_{11}|)(1 - |m_{22}|)$.

Example 4. For HS21, we have $b_1 = d_1 = b_2 = d_2 = 0$. Adding the conditions $a_1 + b_1 = 1/2$ and $a_2 + 3b_2 + 2c_2 + 2d_2 = 1$, so that $a_1 = 1/2$ and $c_2 = (1 - a_2)/2$, we have

$$A_j = \begin{pmatrix} \frac{1}{2} & ja_2 \\ j(\frac{1}{4} + 2c_1) & 1 - \frac{a_2}{2} \end{pmatrix}, \qquad B_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = C_j, \qquad j = \pm 1.$$

The scheme is C^1 if there exists a matrix norm $\|\cdot\|$ such that $\|A_j\| < 1$. For example, sufficient conditions are $0 < a_2 < 4$ and $|a_2| \cdot |1 + 8c_1| < 2 - |2 - a_2|$.

Example 5. Assume that the necessary conditions to get a C^7 interpolant are satisfied, i.e., $a_1 = 243/512, b_1 = 13/512, c_1 = -81/512, d_1 = -3/512$, $a_2 = 405/256, b_2 = 5/256, c_2 = -81/256, d_2 = -1/256$. Then, for $j = \pm 1$,

$$A_j = \begin{pmatrix} \frac{1}{2} & j\frac{205}{128} \\ -j\frac{7}{256} & \frac{51}{256} \end{pmatrix}, \qquad B_j = \begin{pmatrix} j\frac{3}{512} & j\frac{5}{256} \\ j\frac{7}{512} - \frac{3}{1024} & j\frac{13}{256} - \frac{5}{512} \end{pmatrix},$$

$$C_j = \begin{pmatrix} -j\frac{3}{512} & j\frac{5}{256} \\ j\frac{7}{512} + \frac{3}{1024} & -j\frac{13}{256} - \frac{5}{512} \end{pmatrix}.$$

For $\theta = 4$, a numerical evaluation gives $\|A_j\|_\theta + \|B_j\|_\theta + \|C_j\|_\theta = 459/512, j = \pm 1$. Therefore the scheme is C^1.

Example 6. Let a_1 and a_2 be two real numbers. Set $b_1 = 1/2 - a_1$, $c_1 = a_1/2 - 3/8, d_1 = a_1/2 - 1/4, b_2 = 2 - a_2, d_2 = -b_2/2, c_2 = (1 - a_2 - 3b_2 - 2d_2)/2$ then the matrices A_j, B_j, C_j can be written

$$A_j = \begin{pmatrix} \frac{1}{2} & j2 \\ 0 & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & j(2 - a_2) \\ 0 & j(1 - 2a_1) - \frac{2 - a_2}{2} \end{pmatrix},$$

$$C_j = \begin{pmatrix} 0 & j(2 - a_2) \\ 0 & j(2a_1 - 1) - \frac{2 - a_2}{2} \end{pmatrix}.$$

If the condition $|-a_2 + 4a_1| + |4 - a_2 - 4a_1| < 1$ is satisfied, then the scheme is C^1. To get the result, we can use the norm $\| \cdot \|_\theta$ with a θ large enough.

§3. Convergence in a Distributional Sense of HS41

We consider convergence in a distributional sense of the Hermite scheme HS41. Such convergence has already been shown by Derfel, Dyn and Levine [5] in the context of non-Hermite subdivision schemes. There are two basic solutions of our recursive system (1): the first one is the pair (f_0, p_0) with data $f_0(k) = \delta_{k,0}, p_0(k) = 0, k \in \mathbb{Z}$, and the second one is the pair (f_1, p_1) with data $f_1(k) = 0, p_1(k) = \delta_{k,0}, k \in \mathbb{Z}$. These two pairs are important because we can express all the solutions (f, p) of (1) with linear combinations of their translates. For all $n \in \mathbb{N}, j \in \mathbb{Z}$,

$$\begin{aligned} f(j/2^n) &= \sum_{k=-\infty}^{\infty} [f(k)f_0(j/2^n - k) + p(k)f_1(j/2^n - k)], \\ p(j/2^n) &= \sum_{k=-\infty}^{\infty} [f(k)p_0(j/2^n - k) + p(k)p_1(j/2^n - k)]. \end{aligned} \tag{2}$$

Notice that all these sums are finite since the supports of $f_i, p_i, i = 0, 1$ are contained in $[-3, 3]$.

Now, applying relation (2) successively to the pair of functions $(f(x) = f_0(x/2), p(x) = p_0(x/2)/2)$ and then to the pair $(f_1(x/2), p_1(x/2)/2)$ and evaluating the functions f_0, p_0, f_1, p_1 at the half-integers by using (1), we obtain a system of functional equations for f_0, p_0, f_1, p_1 which can be written in a matrix equation as

$$\varphi(x/2) = \sum_{k\in\mathbb{Z}} M_k \varphi(x-k) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{3}$$

where $\varphi(x) = \begin{pmatrix} f_0(x) & p_0(x) \\ f_1(x) & p_1(x) \end{pmatrix}$ and

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} a_1 & -a_2/2 \\ -c_1 & c_2/2 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} a_1 & a_2/2 \\ c_1 & c_2/2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} b_1 & -b_2/2 \\ -d_1 & d_2/2 \end{pmatrix}, \quad M_{-3} = \begin{pmatrix} b_1 & b_2/2 \\ d_1 & d_2/2 \end{pmatrix}, \quad M_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

otherwise.

3.1. Fourier Transform of HS41

Let us begin with a computation without proper justification. We will suppose that the system (3) of functional equations is valid not only when x is a dyadic number, but also when x is an arbitrary real number. We must suppose that f_0, p_0, f_1, p_1 have been extended by continuity on $\mathbb{R}$. Now, we compute the Fourier transform $\hat{f}$ of a function f by $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}\,dx$. Using this Fourier operator on (3), we get

$$\hat{\varphi}(\xi) = A(\xi/2)\hat{\varphi}(\xi/2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{4}$$

where $A(\xi)$ is given by

$$\frac{1}{2}\sum_{k\in\mathbb{Z}} M_k e^{-ik\xi} = \begin{pmatrix} \frac{1}{2} + a_1\cos\xi + b_1\cos 3\xi & \frac{i}{2}(a_2\sin\xi + b_2\sin 3\xi) \\ i(c_1\sin\xi + d_1\sin 3\xi) & \frac{1}{4} + \frac{1}{2}(c_2\cos\xi + d_2\cos 3\xi) \end{pmatrix}.$$

To study the matrix equation in (4), we now look at the matrix product

$$P_n(\xi) = A(\xi/2)A(\xi/4)\cdots A(\xi/2^n). \tag{5}$$

More precisely, we look for conditions on the parameters to get the convergence of the sequence of matrices $P_n(\xi)$. These conditions should be independent of the real or complex value ξ. The study of this sequence for complex values of ξ is motivated by a generalization of Paley-Wiener theorem proposed by Schwartz [8].

Theorem 7. [Schwartz] *Let F be a continuous function on the real axis which is the Fourier transform of a tempered distribution T. Then the support of T is contained in $[-C, C]$ if and only if F may be extended on the complex plane to an analytic entire function of exponential type $\le C$.*

We recall that an entire function $F(z)$ is of exponential type $\le C$ if $\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} \le C$. To study the convergence of the matrix products $P_n(\xi)$, we will also use Proposition 5.2 of Heil and Colella [6].

Proposition 8. *If* $\lim[A(0)]^n$ *exists and is not trivial, then the sequence* $P_n(\xi)$ *converges uniformly on compact sets of* $\mathbb{C}$ *to a continuous matrix-valued function* $P(\xi)$ *whose restriction to the real line has at most polynomial growth at infinity.*

We are now ready for the main result.

Theorem 9. *For* $a_1 + b_1 = 1/2$ *and* $-\frac{5}{2} < c_2 + d_2 \leq \frac{3}{2}$*, for any complex number* ξ*, the sequence of matrices* $P_n(\xi)$ *defined in (5) converges, and the convergence is uniform whenever* ξ *lies in the disk* $|\xi| \leq R$*. As functions of* ξ*, the four components of the limit matrix* $P(\xi)$ *are entire functions of exponential type* ≤ 3.

Proof: $A(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} + \frac{c_2+d_2}{2} \end{pmatrix}$, so that with the hypotheses and the previous proposition, the sequence $P_n(\xi)$ converges on each compact set. As the sequence $P_n(z)$ converges to a matrix $P(z)$, and as the moduli of the components of the matrices $P_n(z)$ are uniformly bounded whenever $|z| \leq R$, the Lebesgue dominated convergence theorem and Cauchy formula give us the proof that all the components of the matrix $P(z)$ are analytic in z.

Finally, let us verify that each element of the matrix $P(z)$ is an entire function of exponential type. Firstly, there exists a real positive number C which depends on the parameters such that for all $z \in \mathbb{C}$, $||A(z)||_\infty \leq Ce^{3|z|}$. Secondly, we know that there exists a real positive number M (depending on the parameters again) such that for all $z \in \mathbb{C}, |z| \leq 1, ||P(z)||_\infty \leq M$.

Let z be a complex number such that $2^n \leq |z| \leq 2^{n+1}$.
As $P(z) = A(z/2)A(z/4)\ldots A(z/2^n)P(z/2^{n+1})$, we obtain the bound

$$||P(z)||_\infty \leq ||P(z/2^{n+1})||_\infty \prod_{k=1}^{n} ||A(z/2^k)||_\infty \leq M \prod_{k=1}^{n} [Ce^{3|z|/2^k}] \leq MC^n e^{3|z|},$$

and thus $\limsup_{|z|\to\infty} \frac{\log ||P(z)||_\infty}{|z|} \leq 3$. Then the functions composing the matrix $P(z)$ are entire functions of exponential type ≤ 3. □

The Schwartz version of the Paley-Wiener theorem implies the following corollary.

Corollary 10. *Set* $a_1+b_1 = 1/2$ *and* $-\frac{5}{2} < c_2+d_2 \leq \frac{3}{2}$*. Then each component function of the limit matrix* $P(z) = \lim P_n(z)$ *is the Fourier transform of a distribution whose support is contained in the interval* $[-3, 3]$.

3.2. Schwartz Distributions Associated with the Scheme

We will connect the computation of Fourier transforms of the previous subsection with the limit matrix $P(\xi)$. This link will come from a sequence of Schwartz matrix distributions. We set

$$T^{(n)} = \frac{1}{2^n} \sum_m \begin{pmatrix} f_0(m/2^n) & p_0(m/2^n) \\ f_1(m/2^n) & p_1(m/2^n) \end{pmatrix} \delta_{m/2^n},$$

where δ_h is the Dirac distribution at point h defined by $\delta_h(\phi) = \phi(h)$. Notice that these sums are finite and that the distributions are compactly supported since the supports of $f_i, p_i, i = 0, 1$ are in $[-3, 3]$.

Now, let us evaluate the Fourier transform of this matrix distribution: $\hat{T}^{(n)}(\xi) = T^{(n)}(e^{-i\xi x})$. Using the system of equations (3), we verify that a simple induction links the sequence of Fourier transforms through the matrix $A(\xi)$. Indeed,

$$\hat{T}^{(n+1)}(\xi) = \frac{1}{2^{n+1}} \sum_m \begin{pmatrix} f_0(m/2^n) & p_0(m/2^n) \\ f_1(m/2^{n+1}) & p_1(m/2^{n+1}) \end{pmatrix} e^{-i\xi m/2^{n+1}}.$$

We substitute $f_0(m/2^{n+1}), p_0(m/2^{n+1}), f_1(m/2^{n+1}), p_1(m/2^{n+1})$ by the right member of the first, second, third, fourth equation of system (3), respectively, with $x = m/2^n$ in the last equation to obtain the recursion

$$T^{(n+1)}(\xi) = A(\xi/2)T^{(n)}(\xi/2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{6}$$

Since $\hat{T}^{(0)}(\xi) = I$, the identity matrix, we get

$$T^{(n)}(\xi) = P_n(\xi) \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}.$$

The sequence of column vectors $\begin{pmatrix} \hat{T}^{(n)}_{11}(\xi) \\ \hat{T}^{(n)}_{21}(\xi) \end{pmatrix}$ converges to the first column of the matrix $P(\xi)$. Now, Schwartz has noticed that the Fourier transform in the space of tempered distributions is a linear continuous transformation and that its inverse is equal to its conjugate ([8] p. 107 of Vol. 2). Therefore the sequences $T^{(n)}_{11}, T^{(n)}_{21}$ converge to the distributions T_0, T_1, respectively, which are the components of the inverse Fourier transform applied to the first column of the matrix $P(\xi)$.

Theorem 11. *For $a_1 + b_1 = 1/2$ and $-\frac{5}{2} < c_2 + d_2 \leq \frac{3}{2}$, the sequences $T^{(n)}_0, T^{(n)}_1$ converge to the distributions T_0, T_1, respectively, which are the components of the inverse Fourier transform applied to the first column of the matrix $P(\xi)$.*

Now, we can prove that the subdivision scheme is always convergent in the space of distributions $\mathcal{D}'(\mathbb{R})$ whenever $a_1 + b_1 = 1/2$ and $-\frac{5}{2} < c_2 + d_2 \leq \frac{3}{2}$. In the following, we use Schwartz notation for the translation operator τ_h, where h is a real number. Given a function ϕ in C_0^∞ and a distribution T, then $\tau_h\phi(x) = \phi(x - h)$ and $\tau_h T(\phi) = T(\tau_h\phi)$.

Theorem 12. *Assume that $a_1 + b_1 = 1/2$ and $-\frac{5}{2} < c_2 + d_2 \leq \frac{3}{2}$. If we build the pair (f, p) by the subdivision scheme (1) from the data $\{y_k, y'_k\}_{k\in\mathbb{Z}}$,* *then the sequence of distributions* $F_n = \frac{1}{2^n}\sum_{m=-\infty}^{\infty} f(m/2^n)\delta_{m/2^n}$ *converges to the distribution* $F = \sum_{k=-\infty}^{\infty}[y_k\tau_{-k}T_0 + y'_k\tau_{-k}T_1]$.

Proof: Let ϕ be a function in C^∞ with support in $[-N, N]$. Then $F_n(\phi) = \frac{1}{2^n}\sum_{m=-N2^n}^{N2^n} f(m/2^n)\phi(m/2^n)$. We use relation (2) to get

$$\begin{aligned} 2^n F_n(\phi) &= \sum_{m=-N2^n}^{N2^n}\sum_{k=-N-1}^{N+1}[y_k f_0(m/2^n - k) + y'_k f_1(m/2^n - k)]\phi(m/2^n) \\ &= \sum_{k=-N-1}^{N+1}\sum_{m=-N2^n}^{N2^n}[y_k f_0(m/2^n) + y'_k f_1(m/2^n)]\phi(m/2^n + k) \\ &= 2^n \sum_{k=-N-1}^{N+1}[y_k T_0^{(n)} + y'_k T_1^{(n)}](\tau_{-k}\phi). \end{aligned}$$

As n tends to infinity, the limit of the sequence $F_n(\phi)$ is

$$\sum_{k=-\infty}^{\infty}[y_k\tau_{-k}T_0 + y'_k\tau_{-k}T_1](\phi). \quad \square$$

Theorem 13. *For* $a_1 + b_1 = 1/2$ *and* $-\frac{3}{2} < c_2 + d_2 \le \frac{1}{2}$, *both sequences of distributions* $T_{12}^{(n)}, T_{22}^{(n)}$ *converge. They converge respectively to the derivatives* T'_0, T'_1 *of the distributions* T_0, T_1 *if and only if* $a_2 + 3b_2 + 2c_2 + 2d_2 = 1$.

Proof: Using the relations (5) and (6), we have

$$\begin{aligned} \begin{pmatrix} \hat{T}_{12}^{(n)}(\xi) \\ \hat{T}_{22}^{(n)}(\xi) \end{pmatrix} &= 2^n P_{n-1}(\xi)A(\xi/2^n)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 2^n P_{n-1}(\xi)\begin{pmatrix} \frac{i}{2}[a_2\sin(\xi/2^n) + b_2\sin(3\xi/2^n)] \\ \frac{1}{4} + \frac{c_2}{2}\cos(\xi/2^n) + \frac{d_2}{2}\cos(3\xi/2^n) \end{pmatrix} \\ &= \frac{i}{2}(a_2 + 3b_2)\xi\begin{pmatrix} \hat{T}_{11}^{(n-1)}(\xi) \\ \hat{T}_{21}^{(n-1)}(\xi) \end{pmatrix} + (\frac{1}{2} + c_2 + d_2)\begin{pmatrix} \hat{T}_{12}^{(n-1)}(\xi) \\ \hat{T}_{22}^{(n-1)}(\xi) \end{pmatrix} \\ &\quad + O(2^{-n}) \\ &= \frac{i}{2}(a_2 + 3b_2)\xi\sum_{k=0}^{n-1}(\frac{1}{2} + c_2 + d_2)^k\begin{pmatrix} \hat{T}_{11}^{(n-1-k)}(\xi) \\ \hat{T}_{21}^{(n-1-k)}(\xi) \end{pmatrix} \\ &\quad + O(n[\max(\frac{1}{2}, \frac{1}{2} + c_2 + d_2)]^n). \end{aligned}$$

If $|\frac{1}{2} + c_2 + d_2| < 1$ which is one of the hypotheses, then the right member of the last vector equation tends to a column vector whose components are respectively $i\xi\frac{(a_2+3b_2)/2}{1/2-c_2-d_2}\hat{T}_0(\xi)$ and $i\xi\frac{(a_2+3b_2)/2}{1/2-c_2-d_2}\hat{T}_1(\xi)$. They are the two respective limits of the sequences $\hat{T}_{12}^{(n)}(\xi), \hat{T}_{12}^{(n)}(\xi)$. These limits are $i\xi\hat{T}_0\xi, i\xi\hat{T}_1(\xi)$ if and only if $\frac{(a_2+3b_2)/2}{1/2-c_2-d_2} = 1$. Using the inverse Fourier transform on each sequence, it is clear that $T_{12}^{(n)}, T_{22}^{(n)}$ converge to the distributions T'_0, T'_1, respectively. $\square$

Let us conclude with a theorem whose proof is similar to the proof of Theorem 12.

Theorem 14. *Suppose that* $a_1 + b_1 = 1/2$, $-\frac{3}{2} \leq c_2 + d_2 \leq \frac{1}{2}$, *and* $a_2 + 3b_2 + 2c_2 + 2d_2 = 1$. *If we build the pair* (f, p) *by the subdivision scheme (1) from the data* $\{y_k, y'_k\}_{k \in \mathbb{Z}}$, *then the sequence of distributions*

$$G_n = \frac{1}{2^n} \sum_{m=-\infty}^{\infty} p(m/2^n) \delta_{m/2^n}$$

converges to the distribution $G = \sum_{k=-\infty}^{\infty} [y_k \tau_{-k} T'_0 + y'_k \tau_{-k} T'_1]$.

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