Chapter 1
Simplex–Splines on the Clough–Tocher Split with Arbitrary Smoothness.

Tom Lyche, Jean-Louis Merrien, and Tomas Sauer

Abstract The space of piecewise polynomials of smoothness r and degree $3r$ is considered on the Clough-Tocher split of a triangle. For any $r \geq 1$ we give a basis of simplex splines for this space, then a Marsden-like identity, which is proved explicitly for $r \leq 3$ and symbolically for $4 \leq r \leq 6$. In addition, generalizing results for $r = 1$, we prove for $r = 2, 3$ a geometry independent bound for the condition number in the infinity norm of this basis, and conditions to connect two triangles with smoothness r.

Keywords: Triangle Mesh, Piecewise polynomials, Interpolation, Simplex Splines, Marsden-like Identity.

1.1 Introduction

Splines over triangulations have applications in several branches of the sciences ranging from finite element analysis, surfaces in computer aided design and other engineering problems, see for example [6, 9, 17]. For many of these applications, piecewise linear C^0 surfaces do not suffice. In some cases, we need smoother elements for modeling, or higher polynomial degrees to increase the approximation order.

In this paper, we are interested in the space of polynomial splines over a triangulation Δ of a polygonal domain Ω of $\mathbb{R}^2$,

$$\mathbb{S}_d^r(\Delta) := \{ f \in C^r(\Omega) : f|_{\mathcal{T}} \in \mathbb{P}_d, \text{ for all } \mathcal{T} \in \Delta \},$$

where $d > r > 0$ are given integers, and $\mathbb{P}_d$ is the space of bivariate polynomials of total degree $\leq d$. The dimension of this finite dimensional vector space is difficult to determine in general [17], but with the restriction $d \geq 3r + 2$ its dimension can be expressed solely in terms of d and r, see [14].

We can use lower degrees if we are willing to split each triangle into a number of subtriangles. The most well known examples are the Clough-Tocher split [5], and the Powell-Sabin 6 and 12 splits [23]. For these splits each triangle is divided into 3,6 and 12 subtriangles, respectively. For material on these splits and B-spline like bases for splines on triangulations see [2, 3, 7, 8, 10, 12, 13, 15–20, 25–37].

Here we consider the Clough-Tocher triangulation Δ_{CT}, see [4, 5], where each triangle in the original triangulation Δ is split into 3 subtriangles by connecting the vertices of each triangle to its barycenter , see Figure 1.1. In [18], Hermite interpolation problems were considered for super-spline subspaces of $\mathbb{S}_{3r+1}^r(\Delta_{CT})$ and $\mathbb{S}_{3r}^r(\Delta_{CT})$ for r even and odd, respectively. It was also stated that these degrees are minimal for global C^r. See also [25]. In [15] stable local bases were constructed for even smaller super-spline subspaces of $\mathbb{S}_{3r+1}^r(\Delta_{CT})$ and $\mathbb{S}_{3r}^r(\Delta_{CT})$.

In this paper, we consider for any $r \in \mathbb{N}$, the spaces $\mathbb{S}_{3r}^r(\triangle)$ on a single triangle $\mathcal{T} = \triangle$ in Δ_{CT} for which we construct a B-spline like basis made out of simplex splines. They constitute a basis for the space since we show that their number agrees with the dimension of the space and that they are linearly independent. For the latter the differentiation formula for simplex splines is used. This extends and generalizes the case $r = 1$ that was considered in [19]. For more on the Clough-Tocher split see [1, 4, 11, 13, 15, 17, 18, 21, 28, 32]. Looking in more

Tom Lyche
Dept. of Mathematics, University of Oslo, PO Box 1053, Blindern, 0316 Oslo, Norway e-mail: tom@math.uio.no

Jean-Louis Merrien
Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France e-mail: jmerrien@insa-rennes.fr

Tomas Sauer
Lehrstuhl für Mathematik mit Schwerpunkt Digitale Bildverarbeitung & FORWISS, Universität Passau, Innstr. 43, D-94032 Passau, Germany e-mail: Tomas.Sauer@uni-passau.de

detail at the cases $r = 2, 3$, corresponding to degrees $d = 6, 9$, we moreover give explicit formulas for connecting two neighboring triangles in a C^r fashion across an edge using Bernstein-Bézier techniques, and give an upper bound for the L_∞ condition number of the basis. This upper bound is independent of the shape of the triangle $\mathcal{T}$. We also give a Marsden-like identity for the reproduction of polynomials which is proved for $r \le 3$ and shown symbolically for $r \le 6$. We conjecture that it holds for any r.

It was shown in [15, 18] that global C^2 continuity cannot be achieved for $d = 6$ for a general triangulation refined by Clough-Tocher splits into Δ_{CT}. However, for r odd it follows, again from [15, 18], that global C^r continuity holds for $d = 3r$. This means in particular that the formulas for C^3 continuity across an edge in Section 1.4.2 can be used to compute with elements in $\mathbb{S}_9^3(\Delta_{CT})$, using the simplex spline basis on each triangle in $\mathbb{S}_9^3(\Delta)$ in the usual Bernstein-Bézier fashion.

Some results in this paper are based on symbolic computation. The first author can provide code in Mathematica for specific results upon request.

The paper is organized as follows. Since it depends heavily on properties of Bernstein polynomials and simplex splines, we recall some well known facts about these functions in the next section. In Section 1.3 we define a collection of simplex splines and show that they constitute a basis for $\mathbb{S}_{3r}^r(\triangle)$, $r \ge 1$. In Section 1.4, we consider the cases of global C^2 and C^3 regularity in more detail. In Subsection 1.5.3, we give a generalization of the barycentric Marsden-like identity for $3 < r$. We add an appendix with some proofs not essential for the results in the paper, but that can be of some use for a reader.

1.2 Preliminaries

In this section we recall some properties of Bernstein polynomials on a triangle and bivariate simplex splines. Here we use the notation $d \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and let $\langle \mathcal{S} \rangle$ denote the **convex hull** of the set $\mathcal{S} \subset \mathbb{R}^2$.

1.2.1 Bernstein Polynomials

For a given nondegenerate triangle $\mathcal{T} := \langle \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\} \rangle \in \mathbb{R}^2$, and $i, j, k \in \mathbb{N}_0$, the Bernstein polynomial $B_{ijk}^d : \mathbb{R}^2 \to \mathbb{R}$ of degree $d := i + j + k \in \mathbb{N}_0$, is defined by

$$B_{ijk}^d(x, y) = B_{ijk}^d(\beta_1, \beta_2, \beta_3) := \frac{d!}{i!j!k!} \beta_1^i \beta_2^j \beta_3^k, \tag{1.1}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ are the **barycentric coordinates** of $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ with respect to $\mathcal{T}$, i. e.,

$$\boldsymbol{x} = \beta_1 \boldsymbol{p}_1 + \beta_2 \boldsymbol{p}_2 + \beta_3 \boldsymbol{p}_3, \quad \beta_1 + \beta_2 + \beta_3 = 1. \tag{1.2}$$

The barycentric form of **Marsden's identity** for Bernstein polynomials is simply the multinomial expansion

$$(u_1\beta_1 + u_2\beta_2 + u_3\beta_3)^d = \sum_{i+j+k=d} u_1^i u_2^j u_3^k B_{ijk}^d(\beta_1, \beta_2, \beta_3), \quad (u_1, u_2, u_3) \in \mathbb{R}^3, \quad \beta_1 + \beta_2 + \beta_3 = 1, \tag{1.3}$$

where in the expression $\sum_{i+j+k=d}$ it is understood that $i, j, k \in \mathbb{N}_0$, which is consistent with the convention that $B_{ijk}^d = 0$ if one of the indices becomes negative.

Taking partial derivatives of order $\nu, \lambda, \kappa \in \mathbb{N}_0$ with respect to u_1, u_2, u_3, respectively, in (1.3) and setting $\boldsymbol{u} = (1, 1, 1)$ we obtain

$$\beta_1^\nu \beta_2^\lambda \beta_3^\kappa = \sum_{i+j+k=d} \gamma_{ijk}(\beta_1^\nu \beta_2^\lambda \beta_3^\kappa) B_{ijk}^d(\beta_1, \beta_2, \beta_3), \qquad \nu + \lambda + \kappa \le d, \quad \gamma_{ijk}(\beta_1^\nu \beta_2^\lambda \beta_3^\kappa) \in \mathbb{R}. \tag{1.4}$$

We find

$$\begin{aligned} 1 &= \sum_{i+j+k=d} B_{ijk}^d(\beta_1, \beta_2, \beta_3) \\ (\beta_1, \beta_2, \beta_3) &= \sum_{i+j+k=d} \boldsymbol{b}_{ijk}^* B_{ijk}^d(\beta_1, \beta_2, \beta_3), \quad \boldsymbol{b}_{ijk}^* = \left(\frac{i}{d}, \frac{j}{d}, \frac{k}{d}\right). \end{aligned} \tag{1.5}$$

The vector $\boldsymbol{b}^*_{ijk}$ is called the **barycentric form** of the **domain point** of B^d_{ijk}. From (1.4) and (1.5) it follows that the elements in the set

$$\mathcal{B}^d := \{B^d_{ijk} : i, j, k \geq 0,\ i + j + k = d\} \tag{1.6}$$

form a **partition of unity basis** for $\mathbb{P}_d$. Indeed, the number of elements $\#\mathcal{B}^d$ of $\mathcal{B}^d$ equals $\binom{d+2}{2}$, the dimension of $\mathbb{P}_d$. We refer to [17] for further properties of B^d_{ijk}.

1.2.2 Bivariate Simplex Splines

For our purpose it is convenient to work with **area normalized bivariate simplex splines** [20] of degree $d \geq 0$ with **knots**

$$K := \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{d+3}\}, \quad \boldsymbol{k}_j \in \mathbb{R}^2, \qquad j = 1, \dots, d+3.$$

We can consider K either as a multiset or as a matrix $K \in \mathbb{R}^{2\times(d+3)}$.

The simplex spline $Q[K] : \mathbb{R}^2 \to \mathbb{R}$, is now defined as $Q[K](\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$ if $\operatorname{area}(\langle K\rangle) = 0$, and otherwise

$$Q[K] := \frac{\operatorname{area}(\mathcal{T})}{\binom{d+2}{2}} M[K], \tag{1.7}$$

where $\mathcal{T}$ is a fixed reference triangle in the original triangulation Δ. Here $\operatorname{area}(\mathcal{S})$ is the area in $\mathbb{R}^2$ of a set $\mathcal{S} \in \mathbb{R}^2$. The function $M[K]$ is a **unit integral bivariate normalized simplex spline**, defined as a linear functional $M[K] : C(\mathbb{R}^2) \to \mathbb{R}$ given by

$$M[K](\varphi) := (d+2)! \int_{\mathcal{S}_{d+2}} \varphi\Big(\sum_{j=1}^{d+3} \boldsymbol{k}_j t_j\Big) \mathrm{d}t_1 \cdots \mathrm{d}t_{d+2}, \qquad \varphi \in C(\mathbb{R}^2), \tag{1.8}$$

with $\mathcal{S}_n := \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1\}$, the unit simplex in $\mathbb{R}^n$, $n \in \mathbb{N}$. If $\operatorname{area}\langle K\rangle = 0$ then $M[K]$ can be identified with a function $M[K] : \mathbb{R}^2 \to \mathbb{R}$, and we write (1.8) in the form

$$\int_{\mathbb{R}^s} M[K](\boldsymbol{x})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = (d+2)! \int_{\mathcal{S}_{d+2}} \varphi\Big(\sum_{j=1}^{d+3} \boldsymbol{k}_j t_j\Big) \mathrm{d}t_1 \cdots \mathrm{d}t_{d+2},, \qquad \varphi \in C(\mathbb{R}^2). \tag{1.9}$$

We mention the following well-known properties of $M[K]$ [22, 24] and $Q[K]$.

1. $Q[K]$ and $M[K]$ are **piecewise polynomials** of degree $d = \#K - 3$ witht support $\langle K\rangle$.
2. **Local smoothness**: Across a **knot line**, which is a line in the complete graph associated with K, we have that $M[K], Q[K] \in C^{d+1-m}$, where m is the number of knots on that knot line, including multiplicities.
3. **Differentiation formula**: For $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ and any choice of $a_1, \dots, a_{d+3}$ such that $\sum_j a_j \boldsymbol{k}_j = \boldsymbol{u}$, $\sum_j a_j = 0$, one has

$$D_{\boldsymbol{u}} M[K] = (d+2) \sum_{j=1}^{d+3} a_j\, M[K \setminus \boldsymbol{k}_j], \quad D_{\boldsymbol{u}} Q[K] = d \sum_{j=1}^{d+3} a_j\, Q[K \setminus \boldsymbol{k}_j], \tag{1.10}$$

   where $D_{\boldsymbol{u}} := u_1 D_1 + u_2 D_2$ and D_1, D_2 denotes partial derivatives. (*A*-**recurrence**)
4. **Recurrence relation**: For any $\boldsymbol{x} \in \mathbb{R}^2$ and any $b_1, \dots, b_{d+3}$ such that $\sum_j b_j \boldsymbol{k}_j = \boldsymbol{x}$, $\sum_j b_j = 1$, one has

$$M[K](\boldsymbol{x}) = \frac{d+2}{d} \sum_{j=1}^{d+3} b_j M[K \setminus \boldsymbol{k}_j](\boldsymbol{x}), \quad Q[K](\boldsymbol{x}) = \sum_{j=1}^{d+3} b_j Q[K \setminus \boldsymbol{k}_j](\boldsymbol{x}). \tag{1.11}$$

   (*B*-**recurrence**)
5. **Knot insertion formula**: For any $\boldsymbol{y} \in \mathbb{R}^2$ and any $c_1, \dots, c_{d+3}$ such that $\sum_j c_j \boldsymbol{k}_j = \boldsymbol{y}$, $\sum_j c_j = 1$, one has

$$M[K] = \sum_{j=1}^{d+3} c_j M[K \cup \boldsymbol{y} \setminus \boldsymbol{k}_j], \quad Q[K] = \sum_{j=1}^{d+3} c_j Q[K \cup \boldsymbol{y} \setminus \boldsymbol{k}_j]. \tag{1.12}$$

   (*C*-**recurrence**)

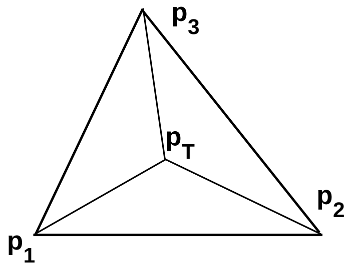


Fig. 1.1 The Clough–Tocher split, $\mathcal{T} = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$, $\mathcal{T}_1 := \langle \boldsymbol{p_T}, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$, $\mathcal{T}_2 := \langle \boldsymbol{p_T}, \boldsymbol{p}_3, \boldsymbol{p}_1 \rangle$ and $\mathcal{T}_3 := \langle \boldsymbol{p_T}, \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$

6. **Degree zero**: For $K = \{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\}$

$$\begin{aligned} M[K](\boldsymbol{x}) &:= \frac{1}{\operatorname{area}(\langle K \rangle)}, \quad Q[K](\boldsymbol{x}) := \frac{\operatorname{area}(\mathcal{T})}{\operatorname{area}(\langle K \rangle)}, \ \boldsymbol{x} \in \langle K \rangle^o, \\ M[K](\boldsymbol{x}) &:= Q[K](\boldsymbol{x}) = 0, \ \boldsymbol{x} \notin \langle K \rangle, \end{aligned} \tag{1.13}$$

where $\mathcal{S}^o$ is the interior of the set $\mathcal{S}$. The values of $M[K]$ and $Q[K]$ on the boundary of $\langle K \rangle$ has to be dealt with separately, see below.

We refer to [22, 24] for further properties of $M[K]$.

1.3 The Clough-Tocher split and a basis for $\mathbb{S}^r_{3r}$

Given a nondegenerate triangle $\mathcal{T}$ in $\mathbb{R}^2$, we connect the vertices $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ to the barycenter $\boldsymbol{p_T} := (\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)/3$. With this construction, known as the **Clough–Tocher split** $\triangle$, we obtain three subtriangles $\mathcal{T}_1 := \langle \boldsymbol{p_T}, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$, $\mathcal{T}_2 := \langle \boldsymbol{p_T}, \boldsymbol{p}_3, \boldsymbol{p}_1 \rangle$ and $\mathcal{T}_3 := \langle \boldsymbol{p_T}, \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$, see Figure 1.1.

We consider the spline space $\mathbb{S}^r_d(\triangle)$ with respect to these three subtriangles on $\mathcal{T}$. To obtain a unique function value at each point in $\mathcal{T}$ we associate the half open edges

$$\langle \boldsymbol{p}_i, \boldsymbol{p_T} \rangle := \{(1-t)\boldsymbol{p}_i + t\boldsymbol{p_T} : 0 \le t < 1\}, \quad i = 1, 2, 3,$$

to the three subtriangles of $\mathcal{T}$ as follows

$$\langle \boldsymbol{p}_1, \boldsymbol{p_T} \rangle \in \mathcal{T}_2, \quad \langle \boldsymbol{p}_2, \boldsymbol{p_T} \rangle \in \mathcal{T}_3, \quad \langle \boldsymbol{p}_3, \boldsymbol{p_T} \rangle \in \mathcal{T}_1, \tag{1.14}$$

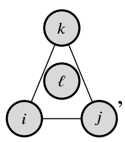
and we somewhat arbitrarily associate the point $\boldsymbol{p_T}$ to $\mathcal{T}_2$.

The dimension of the space $\mathbb{S}^r_d(\triangle)$ is given in the following proposition. It follows from Theorem 9.3 in [17] with $n = m_v = 3$.

Proposition 1 *We have*

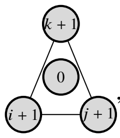
$$\dim \mathbb{S}^r_d(\triangle) = \binom{r+2}{2} + 3\binom{d-r+1}{2} + \sum_{j=1}^{d-r} (r+1-2j)_+ \quad \blacksquare \tag{1.15}$$

We focus on a special collection of simplex splines on the Clough–Tocher split of a single triangle in the following way: for integers i, j, k, ℓ, we consider the simplex spline $Q\left[\boldsymbol{p}_1^{\{i\}}, \boldsymbol{p}_2^{\{j\}}, \boldsymbol{p}_3^{\{k\}}, \boldsymbol{p}_{\mathcal{T}}^{\{\ell\}}\right]$, where $\boldsymbol{p}_j^{\{m\}}$, means that the vertex $\boldsymbol{p}_j$ has multiplicity m, i. e., is repeated m times. For brevity, we will also use two alternative notations, a more compact, and a more illustrative one, namely,

$$Q\left[\boldsymbol{p}_1^{\{i\}}, \boldsymbol{p}_2^{\{j\}}, \boldsymbol{p}_3^{\{k\}}, \boldsymbol{p}_{\mathcal{T}}^{\{\ell\}}\right] =: \triangle[i, j, k; \ell] =: \triangle_{ijk\ell} =:$$ 

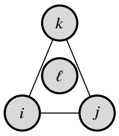
$$, \quad i, j, k, \ell \in \mathbb{N}_0. \tag{1.16}$$

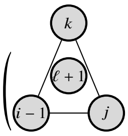
We note that $\triangle_{ijk\ell}$ is a simplex spline of degree $d = i + j + k + \ell - 3$, and by the local smoothness property has smoothness $d + 1 - i - \ell$ across the knotline $\langle \boldsymbol{p}_1, \boldsymbol{p}_T \rangle$ for $\ell > 0$. In this notation, the Bernstein polynomials $B_{ijk}(u, v, w) := \frac{(i+j+k)!}{i!j!k!} u^i v^j w^k$ of degree $d = i + j + k$ have the form

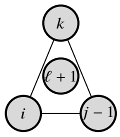
$$B_{ijk} = \triangle_{i+1,j+1,k+1,0} =$$ 

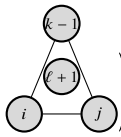
$$, \qquad i + j + k = d. \tag{1.17}$$

We will use the more graphic form on the right hand side of (1.16) whenever possible to make the basic ideas more accessible, but it is convenient to use the more compact $\triangle$ notations in computations. For $i, j, k \in \mathbb{N}$, the *knot insertion formula* (1.12) for the insertion of a knot at the barycenter can be written



$= \frac{1}{3}\Bigg($ 

$+$ 

$+$ 

$\Bigg)$. (1.18)

Definition 1 The number $\mu := d + 1 - r$ is called the maximum *multiplicity* of an interior knot line of the simplex spline $\triangle_{ijk\ell}$.

Some obvious properties for simplex splines to be in $\mathbb{S}^r_d(\triangle)$ are listed in the following lemma.

Lemma 1 *If* $\triangle_{ijk\ell} \in \mathbb{S}^r_d(\triangle)$ *then*

1. $i + j + k + \ell = d + 3 = \mu + r + 2$,
2. *if, in addition,* $\ell \neq 0$ *then regularity* C^r *implies* $\max\{i, j, k\} + \ell \leq \mu$. ■

Next, let us turn to splines of degree $3r$, and focus on specific elements of $\mathbb{S}^r_{3r}(\triangle)$ that will turn out to be useful. The three types are categorized by whether $\max\{i, j, k\} + \ell$ equals μ, exceeds it or is strictly less than μ, where the second case occurs only for $\ell = 0$.

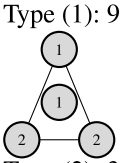
Definition 2 Let Σ^r_{3r} denote the set of elements $\triangle_{ijk\ell}$ of $\mathbb{S}^r_{3r}(\triangle)$ that consists of the elements of the following three types:

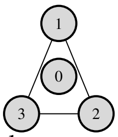
Type (1): $\max\{i, j, k\} + \ell = \mu$ and $\min\{i, j, k\} \geq 1$,
Type (2): $\ell = 0$ and $\max\{i, j, k\} > \mu$,
Type (3): $\max\{i, j, k\} + \ell < \mu$ and $\min\{i, j, k\} = 1$.

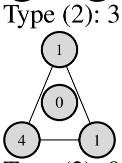
Remark 1 The types are symmetric with respect to i, j, k, i. e., all subclasses are closed under permutation of the indices.

For $r = 1, 2, 3, 4$, Definition 2 results in the following set of splines.

Example 1 For $r = 1$, $d = 3$, $\mu = 3$, $\dim \mathbb{S}^1_3(\triangle) = 12$ we have

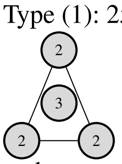
1. Type (1): 9 elements


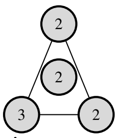


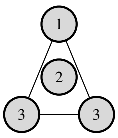
and symmetries
2. Type (2): 3 elements


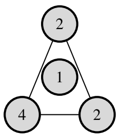
and symmetries,
3. Type (3): 0 elements.

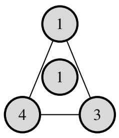
Example 2 For $r = 2$, $d = 6$, $\mu = 5$, $\dim \mathbb{S}^2_6(\triangle) = 37$ the different types look as follows:

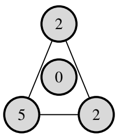
1. Type (1): 25 elements


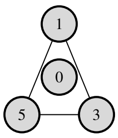






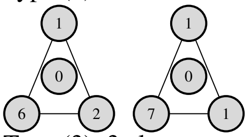






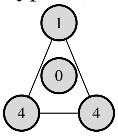
and symmetries,

2. Type (2) 9 elements



and symmetries

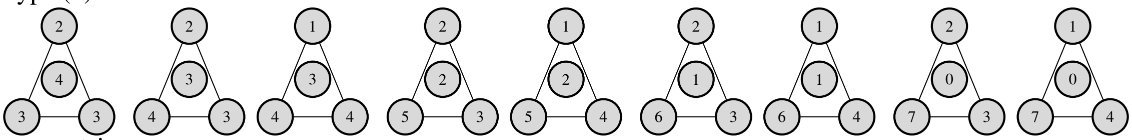
3. Type (3): 3 elements



and symmetries.

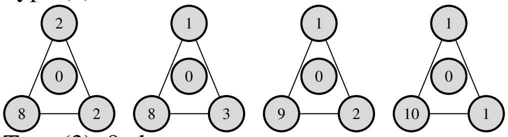
Example 3 In the case $r = 3$, $d = 9$, $\mu = 7$, $\dim \mathbb{S}_9^3(\triangle) = 75$ we get

1. Type (1): 48 elements



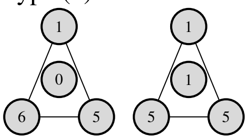
and symmetries,

2. Type (2): 18 elements



and symmetries

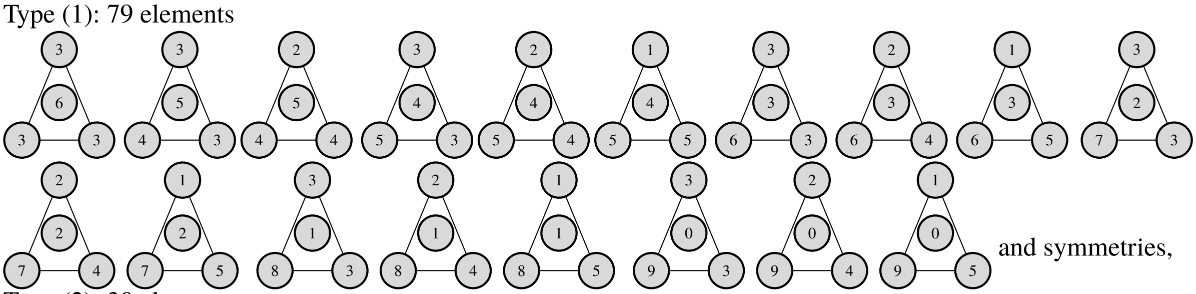
3. Type (3): 9 elements



and symmetries

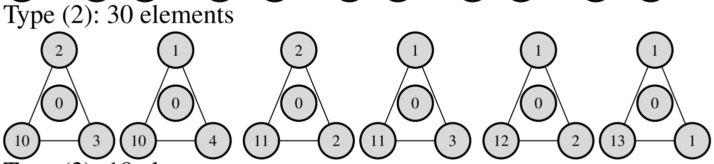
Example 4 For $r = 4$, $d = 12$, $\mu = 9$, $\dim \mathbb{S}_{12}^4(\triangle) = 127$, the elements look as follows:

1. Type (1): 79 elements



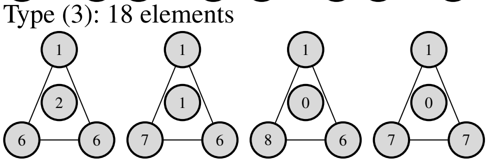
and symmetries,

2. Type (2): 30 elements



and symmetries

3. Type (3): 18 elements



and symmetries

Remark 2 Some Bernstein polynomials are **not** of any of the three types and thus are not in the basis. A Bernstein polynomial in the basis can be of Type (1), (2) or (3), see Figure 1.2 and the previous examples.

One central result of this paper, stated in Theorem 1, is that Σ_{3r}^r is in fact a *basis* for $\mathbb{S}_{3r}^r(\triangle)$. The proof of this fact will consist of showing that Σ_{3r}^r is a subset of $\dim(\mathbb{S}_{3r}^r(\triangle))$ linearly indenpendent elements of the space $\mathbb{S}_{3r}^r(\triangle)$.

To count the number of elements of Σ_{3r}^r, we start with some bounds of ℓ with respect to the different types of functions in Σ_{3r}^r.

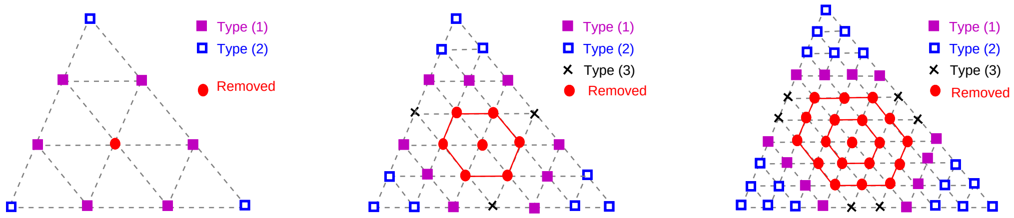


Fig. 1.2 Domain points of the Bernstein polynomials used in the CTS basis, left: $r = 1, d = 3$, middle: $r = 2, d = 6$, right: $r = 3, d = 9$. "Removed" means that the respective Bernstein polynomials are of none of the Types (1), (2) or (3).

Lemma 2 *For Type (1) we have $\ell \le d - 3r/2$ while for Type (3) $\ell \le d - 2r - 2$ holds.*

Proof For a Type (1) element, assume that $i + \ell = \mu$, then $j + \ell \le \mu$ and $k + \ell \le \mu$, hence $i + j + k + 3\ell \le 3\mu$. Since $i + j + k + \ell = d + 3$ and $\mu = d - r + 1$, we thus deduce that $d + 3 + 2\ell \le 3d + 3 - 3r$ or $\ell \le d - 3r/2$.

For Type (3), we assume that $k = 1$, hence $i + \ell \le \mu - 1$ and $j + \ell \le \mu - 1$, then $i + j + k + 2\ell \le 2\mu - 1$. Since $i + j + k + \ell = d + 3$ and $\mu = d - r + 1$, it follows that $d + 3 + \ell \le 2d - 2r + 2 - 1$ or $\ell \le d - 2r - 2$.
■

Next we prove that the sum of the numbers of elements of the three types is exactly the dimension of $\dim \mathbb{S}^r_{3r}(\triangle)$.

Proposition 2 *For $r \ge 1$ and $d = 3r$, we have the following table according to the parity of r.*

r	$2s,\ s > 0$	$2s + 1,\ s \ge 0$
d	$6s$	$6s + 3$
$\dim \mathbb{S}^r_d(\triangle)$	$27s^2 + 9s + 1$	$3(9s^2 + 12s + 4)$
$\#Type(1)$	$3s(5s + 3) + 1$	$3(s + 1)(5s + 3)$
$\#Type(2)$	$3s(2s + 1)$	$3(s + 1)(2s + 1)$
$\#Type(3)$	$3s(2s - 1)$	$3s(2s + 1)$

(1.19)

Hence,

$$\#Type(1) + \#Type(2) + \#Type(3) = \dim \mathbb{S}^r_{3r}(\triangle), \qquad r \ge 1. \tag{1.20}$$

Proof We begin with r of *even* parity, i. e., $r = 2s$, $d = 3r = 6s$, $\mu = 4s + 1$, $i + j + k + \ell = 6s + 3$ and count the basis elements of different types.

1. Type (1): According to Lemma 2 we have $\ell \le d - 3r/2 = 3s$ and a generic element of Type (1) is of the form

$$\triangle[i = s + p + 1, j, k = 2s + 2 - j; 3s - p], \qquad 1 \le j \le i, 1 \le k \le i, \tag{1.21}$$

from which it follows that $s - p + 1 \le j \le 2s + 1$ and $p = 0, \ldots, 3s$. We count the number of elements with respect to p:

a. For $p = 0$: 1 element.
The only possible choice is $\triangle[s + 1, s + 1, s + 1; 3s]$.

b. For $1 \le p \le s$: $6p$ elements.
The elements $\triangle[i, j, k; 3s - p]$ are of the form

$$\triangle[s + p + 1, s - p + 1 + q, s + p + 1 - q; 3s - p], \qquad q = 0, \ldots, 2p - 1, \tag{1.22}$$

i. e., $i = s + p + 1$, $j = s - p + 1 + q$, $s + p + 1 - q$, and with the permutations $[i, j, k]$, $[j, k, i]$ and $[k, j, i]$, which gives 3 elements for any q in (1.22). We notice that in two cases we obtain the same three elements up to the permutation, namely, for $q = 0$ the values $i = k = s + p + 1$, $j = s - p + 1$, and for $q = 2p$ the values $i = j = s + p + 1$, $k = s - p + 1$. In the other cases, $0 < q < 2p$, we have $j = s - p + 1 + q < i = s + p + 1$ and $k = s + p + 1 - q < i = s + p + 1$, so that the elements are different.

c. For $s + 1 \le p \le 3s$: $6s + 3$ elements
consisting of

$$\triangle[s + p + 1, q + 1, 2s - q + 1; 3s - p], \qquad q = 0, \ldots, 2s, \tag{1.23}$$

and again the permutations $[i, j, k]$, $[j, k, i]$, $[k, j, i]$.

Hence the total number of elements of Type (1) is

$$1+\sum_{p=1}^{s}6p+\sum_{p=s+1}^{3s}(6s+3)=1+3s(s+1)+2s(6s+3)=1+3s(5s+3)$$

as listed in (1.19).

2. Type (2) requires $\ell=0$ by definition and the generic elements of the form

$$\triangle[i=4s+2+p,j,k=s+1-j-p;0],\qquad 1\le j\le i,\ i\le k\le i, \tag{1.24}$$

with the additional constraint $i\ge\mu+1=4s+2$ that leads to $1\le j\le 2s-p$, $p=0,\dots,2s-1$. This gives $3(2s-p)$ elements

$$\triangle[4s+2+p,q+1,2s-p-q;0],\qquad q=0,\dots,2s-p-1, \tag{1.25}$$

and the respective permutations so that the total number of elements of Type (2) is

$$\sum_{p=1}^{2s-1}3(2s-p)=\sum_{q=1}^{2s}q=3s(2s+1)$$

in this case.

3. Type (3): up to symmetry we can assume that $k=1$ and, by Lemma 2 that $\ell\le d-2r-2=2s-2$. The generic element is of the form

$$\triangle[i,j=4s+4+p-i,k=1;\ell=2s-2-p],\qquad i+\ell\le\mu-1=4s,\ j+\ell\le 4sp=0,\dots,2s-2, \tag{1.26}$$

since $i+j+k+\ell=d+3=6s+3$. Hence, for $0\le p\le 2s-2$ we get $3(p+1)$ elements

$$\triangle[2s+2+p-q,2s+2+q,1;2s-2-p],\qquad q=0,\dots,p, \tag{1.27}$$

and the permutations $[i,j,k]$, $[j,k,i]$, $[k,j,i]$, leading to a total of

$$\sum_{p=0}^{2s-2}3(p+1)=3s(2s-1)$$

elements of Type (3).

In the case of odd parity, i. e., $r=2s+1$, $d=3r=6s+3$, $\mu=4s+3$, $i+j+k+\ell=6s+6$, we proceed in the same way and distinguish by types.

1. For Type (1) we have the bound $\ell\le 3s+1$ and the generic element

$$\triangle[i=s+p+2,j,k=2s+3-j;3s+1-p],\qquad 1\le j\le i\ 1\le k\le i,\ p=0,\dots,3s+1. \tag{1.28}$$

Again, Type (1) request the distinction of several cases according to p.

a. For $0\le p\le s$: $6p+3$ elements
The generic elements are

$$\triangle[s+p+2,s+1-p+q,s+2+p-q;3s+1-p],\qquad q=0,\dots,2p, \tag{1.29}$$

and the permutations $[i,j,k]$, $[j,k,i]$, $[k,j,i]$. Again, we notice that we obtain the same three elements up to the permutations for $q=0$, namely $i=k=s+p+2$, $j=s-p+1$, and for $q=2p+1$, namely $i=j=s+p+2$, $k=s-p+1$, respectively. For $0<q<2p$, on the other hand, we have $j=s+1-p+q<i=s+p+2$ and $k=s+2+p-q<i=s+p+2$ so that all the elements are different again, just like in the case of even r.

b. For $s+1\le p\le 3s+1$: $6(s+1)$ elements
based on the generic element

$$\triangle[s+p+2,q+1,2s-q+2;3s+1-p],\qquad q=0,\dots,2s+1, \tag{1.30}$$

and its permutations.

Therefore, the total number of elements of Type (1) is

$$\sum_{p=0}^{s}(6p+3)+\sum_{p=s+1}^{3s+1}6(s+1)=3(s+1)(5s+3).$$

2. Type (2) again requests $\ell = 0$ and leads to $3(2s+1-p)$ elements based on the generic element

$$\triangle[4s+4+p, q+1, 2s+1-p-q; 0], \qquad q=0,\dots,2s-p, \tag{1.31}$$

and its permutations, so that the total number of elements of Type (2) is

$$\sum_{p=0}^{2s}3(2s+1-p)=3(s+1)(2s+1).$$

3. For Type (3) we again assume that $k=1$, note $\ell \le d-2r-2=2s-1$ and obtain $3(p+1)$ elements from the generic element

$$\triangle[2s+3+p-q, 2s+3+q, 1; 2s-1-p], \qquad q=0,\dots,p \tag{1.32}$$

and its permutations totalling up to

$$\sum_{p=0}^{2s-1}3(p+1)=3s(2s+1)$$

elements of Type (3). □

Having completed the table in (1.19) the claim (1.20) follows from summing up the columns of the table.
■

Theorem 1 *Σ_{3r}^{r} is a basis of $\mathbb{S}_{3r}^{r}$.*

We prove Theorem 1 by verifying in Proposition 3 that the elements of Σ_{3r}^{r} are linearly independent. Since we already know from Proposition 2 that $\#\Sigma_{3r}^{r}=\dim\mathbb{S}_{3r}^{r}$, this indeed shows that they are a basis of the spline space. Consequently, Σ_{3r}^{r} spans the space of all simplex splines contained $\mathbb{S}_{3r}^{r}$. We give an independent proof of this fact in Proposition 8 in the appendix as it may be of independent interest and motivates the classification of the simplex splines according to the three types.

To prove linear independence, we need the following technical tool concerning particular derivatives of simplex splines.

Lemma 3 *For $i>0$ We have that*

$$D_{\boldsymbol{p}_1-x}\triangle[i,j,k;\ell](x)=d\left(\triangle[i-1,j,k;\ell](x)-\triangle[i,j,k;\ell](x)\right), \tag{1.33}$$

while

$$\begin{aligned}D_{\boldsymbol{p}_1-x}\triangle[0,j,k;\ell](x)=&d\Big(3\,\triangle[0,j,k;\ell-1](x)\\&-\triangle[0,j-1,k;\ell](x)-\triangle[0,j,k-1;\ell](x)-\triangle[0,j,k;\ell](x)\Big).\end{aligned} \tag{1.34}$$

Proof Write $x=\sum\alpha_j\boldsymbol{p}_j$, with $\sum\alpha_j=1$. Then the derivative formula and the recurrence yield that

$$\begin{aligned}D_{\boldsymbol{p}_1-x}\triangle[i,j,k;\ell](x)&=d\left((1-\alpha_1)\triangle[i-1,j,k;\ell](x)-\alpha_2\triangle[i,j-1,k;\ell](x)-\alpha_3\triangle[i,j,k-1;\ell](x)\right)\\&=d\,\triangle[i-1,j,k;\ell](x)-d\,\left(\alpha_1\triangle[i-1,j,k;\ell](x)+\alpha_2\triangle[i,j-1,k;\ell](x)+\alpha_3\triangle[i,j,k-1;\ell](x)\right)\\&=d\left(\triangle[i-1,j,k;\ell](x)-\triangle[i,j,k;\ell](x)\right),\end{aligned}$$

which is (1.33). For the second identity we note that $\boldsymbol{p}_{\mathcal{T}}=\frac{1}{3}(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)$ implies $\boldsymbol{p}_1=3\boldsymbol{p}_{\mathcal{T}}-\boldsymbol{p}_2-\boldsymbol{p}_3$ and hence, writing $x=\alpha_{\mathcal{T}}\boldsymbol{p}_{\mathcal{T}}+\alpha_2\boldsymbol{p}_2+\alpha_3\boldsymbol{p}_3$ with $\sum\alpha_j=1$, we obtain that

$$\begin{aligned}&D_{\boldsymbol{p}_1-x}\triangle[0,j,k;\ell](x)\\&\quad=d\left((3-\alpha_{\mathcal{T}})\triangle[0,j,k;\ell-1](x)-(1+\alpha_2)\triangle[0,j-1,k;\ell](x)-(1+\alpha_3)\triangle[0,j,k-1;\ell](x)\right)\end{aligned}$$

which can be recombined as above to yield (1.34).
■

Since for $i,j,k,l \in \mathbb{N}_0$, and $\boldsymbol{x} \in \mathcal{T}$ with barycentric coordinates β_1,β_2,β_3 with respect to $\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3$, we have

$$\begin{aligned}
\triangle[0,j+1,k+1;\ell+1](\boldsymbol{x}) &= \frac{(j+k+\ell)!}{j!k!\ell!}(\beta_2-\beta_1)^j(\beta_3-\beta_1)^k(3\beta_1)^\ell, \quad \boldsymbol{x} \in \mathcal{T}_1,\\
\triangle[i+1,0,k+1;\ell+1](\boldsymbol{x}) &= \frac{(i+k+\ell)!}{i!k!\ell!}(\beta_1-\beta_2)^i(\beta_3-\beta_2)^k(3\beta_2)^\ell, \quad \boldsymbol{x} \in \mathcal{T}_2,\\
\triangle[i+1,j+1,0;\ell+1]\boldsymbol{x}) &= \frac{(i+j+\ell)!}{i!j!\ell!}(\beta_1-\beta_3)^i(\beta_2-\beta_3)^j(3\beta_3)^\ell, \quad \boldsymbol{x} \in \mathcal{T}_3,\\
\triangle[i+1,j+1,k+1;0](\boldsymbol{x}) &= \frac{(i+j+k)!}{i!j!k!}\beta_1^i\beta_2^j\beta_3^k = B^d_{ijk}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{T},
\end{aligned} \tag{1.35}$$

we observe that the formula (1.34) in fact corresponds to taking a partial derivative with respect to β_1 in (1.35).

Proposition 3 *The elements of* Σ^r_{3r} *are linearly independent.*

Proof Denote by

$$\mathcal{I}^r := I(\Sigma^r_{3r}) = \left\{(i,j,k,\ell) : \triangle[i,j,k;\ell] \in \Sigma^r_{3r}\right\} \tag{1.36}$$

the set of all knot multiplicities of splines in Σ^r_{3r}. Now assume that there exist coefficients $a_{ijk\ell}$ such that

$$s := \sum_{(i,j,k,\ell)\in\mathcal{I}^r} a_{ijk\ell}\,\triangle[i,j,k;\ell] = 0.$$

On the boundary $\langle \boldsymbol{p}_2,\boldsymbol{p}_3\rangle$ we have $\triangle[i,j,k;\ell] \neq 0$ if and only if $i=1$ and $\ell=0$ and the splines $\triangle[1,j,k;0]$ reduce to univariate Bernstein polynomials that can with $\ell=0$, be classified as follows:

1. Type (1): $\triangle[1,\mu-\ell,\ell+r+1;0]$,
2. Type (2): $\triangle[1,\mu+m,r+1-m;0]$, $1 \le m \le r$,
3. Type (3): $\triangle[1,\mu-\ell-m,\ell+r+1+m;0]$, $1 \le m \le \frac{\mu-r-1}{2}-\ell$, □

together with their symmetric elements where j and k are interchanged. Recall that if these symmetries coincide they are considered as only one element in Σ^r_{3r}. These Bernstein polynomials are linearly independent within the same type by construction and between types since the maximal multiplicity is $=\mu-\ell$ in Type (1), $>\mu-\ell$ in Type (2) and $<\mu-\ell$ in Type (3). Therefore, restricting s to the boundary $\langle \boldsymbol{p}_2,\boldsymbol{p}_3\rangle$, it follows that $a_{ijk\ell}=0$ for $(i,j,k,\ell)=(1,j,k,0)$. Considering the other boundaries of $\mathcal{T}$, we can thus conclude that $a_{ijk\ell}=0$ whenever $\min\{i,j,k\}=1$ and $\ell=0$.

Starting from this observation, we prove by induction on $m=1,2,\dots$ that

$$a_{ijk\ell} = 0, \qquad \min\{i,j,k\}+\ell = m, \tag{1.37}$$

where the case $m=1$ has been treated in the first part of this proof. We will treat the case $m=2$ explicitly as the general procedure will become clear by then. We assume that i is the minimal value, consider the identity

$$0 = D_{\boldsymbol{p}_1-x}s(x), \qquad x \in \langle \boldsymbol{p}_2,\boldsymbol{p}_3\rangle$$

and find by Lemma 3 that there are only two types of splines which are nonzero on the boundary. The first is

$$D_{\boldsymbol{p}_1-x}\triangle[2,j,k;0](x) = \triangle[1,j,k;0](x) - \triangle[2,j,k;0](x)$$

which coincides with $\triangle[1,j,k;0](x)$ on $\langle \boldsymbol{p}_2,\boldsymbol{p}_3\rangle$. The second is

$$D_{\boldsymbol{p}_1-x}\triangle[1,j,k;1](x) = \triangle[0,j,k;1](x) - \triangle[1,j,k;1](x),$$

which by (1.35) coincides with $\triangle[0,j,k;1] = \triangle[1,j,k;0]$ on $\langle \boldsymbol{p}_2,\boldsymbol{p}_3\rangle$. Again, these splines are linearly independent within the types by construction and across the types by the different values of the maximal multiplicity, and (1.37) for $m=2$ follows by considering all three faces of $\mathcal{T}$ by a symmetry argument.

The general induction step proceeds in exactly this way by assuming that $i=\min\{i,j,k\}$ and $i+\ell=m+1$ and applying (1.33) i times we find from (1.35) with $\boldsymbol{x}=(\beta_1,\beta_2,\beta_3)$ and $\ell=m+1-i$

$$D^i_{\boldsymbol{p}_1-x}\triangle[i,j,k;\ell](x) = K\triangle[0,j,k;\ell](x) = \frac{(j+k+\ell)!}{j!k!\ell!}(\beta_2-\beta_1)^j(\beta_3-\beta_1)^k(3\beta_1)^\ell$$

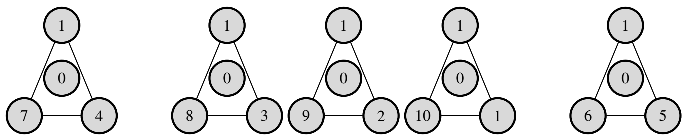
for some connstant K. Differentiating ℓ times with respect to β_1 and setting $\beta_1=0$ we find

$$\frac{\partial^\ell}{\partial \beta_1^\ell} D^i_{\boldsymbol{p}_1 - x} \triangle[i, j, k; \ell](x) = K \beta_2^j \beta_3^k,$$

where again K is some nonzero constant. where the admissible values for j, k lead to linearly independent polynomials on the boundary. This completes the proof of Proposition 3.
■

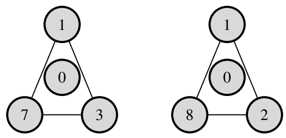
Example 5 We illustrate the elimination procedure of Proposition 3 for the case $r = 3$.

The elimination proceudre starts by considering the Bernstein polynomials

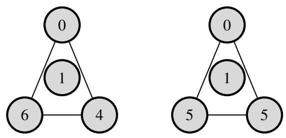


on the "lower" edge of the triangle where alle other simplex splines from our list vanish. Note that the three types are then distinguished by whether the maximum equals $\mu = 7$ (the element on the left), exceeds this values (the three elements in the middle) or is strictly smaller (the element on the right). Therefore, the coefficients of these splines and all their symmetries have to vanish which deals with the first "ring" of coefficients on the boundary.

Applying the differential operator $D_{\boldsymbol{p}_1 - x}$ once, gives us the Bernstein polynomials

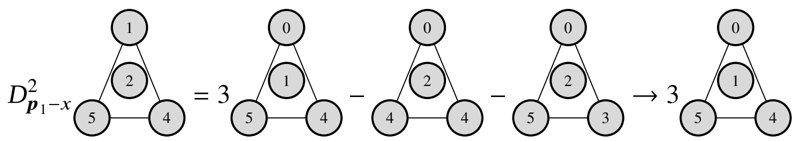


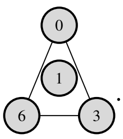
of Type (1) and (2). As the list for $r = 4$ shows, Type (3) elements are not excluded in principle. In addition, we get the splines



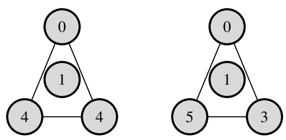
which also reduce to nonvanishing univariate Bernstein polynomials on the boundary. This finishes off all elements with $\min\{i, j, k\} + \ell = 2$ and especially all elements of the Types (2) and (3) for $r = 2$.

The application of $D^2_{\boldsymbol{p}_1 - x}$ gives us

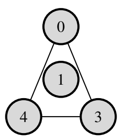


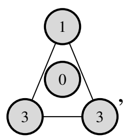
since the two Berstein polynomias which are subtracted vanish on the boundary. The other nonvanishing element is  .

Now it should be clear how the process is completed: $D^3_{\boldsymbol{p}_1 - x}$ extracts the nonzero polynomials



together with their symmetries and applying $D^4_{\boldsymbol{p}_1 - x}$ and $D^5_{\boldsymbol{p}_1 - x}$ we get



and


,

respectively. Observe that in each step of the process always Bernstein polynomials of the same fixed degree are considered on the boundary.

1.4 Marsden-like Identity, Domain points, Stability and C^r–connection

A Marsden-like identity allows us to derive *explicit* formulas for the representation of polynomials of degree up to $3r$ in the simplex spline basis of $\mathbb{S}^r_{3r}(\triangle)$. Even if we have identified a basis of $\mathbb{S}^r_{3r}(\triangle)$, it turns out that for the partition of unity and the Marsden-like identity alone, i.e., for the generation of polynomials and espcially the constant function, we do not need all the elements of that basis. Hence, instead of having redundancy or a null component in the sums, we remove one further element according to the following definition.

Definition 3 Depending on the parity of r, we define the following sets and spaces:

1. If $r = 2s + 1$ with $s \geq 0$, then
$$\bar{\Sigma}^r_{3r} := \Sigma^r_{3r}, \qquad \bar{\mathbb{S}}^r_{3r}(\triangle) = \mathbb{S}^r_{3r}(\triangle).$$
2. If $r = 2s$ then
$$\bar{\Sigma}^r_{3r} := \Sigma^r_{3r} \setminus \left\{\triangle(s+1, s+1, s+1; 3s)\right\}, \qquad \bar{\mathbb{S}}^r_{3r}(\triangle) := \operatorname{span} \bar{\Sigma}^r_{3r}.$$

In analogy with (1.36), the set of indices (i, j, k, ℓ) of the basis $\bar{\Sigma}^r_{3r}$ will be written as $\bar{\mathcal{I}}^r$. Moreover, for the Bernstein polynomials in $\bar{\Sigma}^r_{3r}$ (cf. Figure 1.2), we define the index sets

$$\begin{aligned} \mathcal{I}^r_{\text{removed}} &:= \{(i, j, k) : i + j + k = 3r,\ \max\{i, j, k\} \leq 2r - 1,\ \min\{i, j, k\} \geq 1\}, \\ \mathcal{I}^r_{\text{Bernstein}} &:= \{(i, j, k) : i, j, k \in \mathbb{N}_0,\ i + j + k = 3r\} \setminus \mathcal{I}^r_{\text{removed}}. \end{aligned} \tag{1.38}$$

It immediately follows that

$$\#\mathcal{I}^r_{\text{Bernstein}} = \frac{3}{2} r(r+5), \quad \#\mathcal{I}^r_{\text{removed}} = 3r(r-1) + 1 \quad \#\mathcal{I}^r_{\text{Bernstein}} + \#\mathcal{I}^r_{\text{removed}} = \binom{3r+2}{2} = \dim \mathbb{P}_{3r}. \tag{1.39}$$

We now consider the cases $r = 2, 3$ in more detail. We prove, using knot insertion, a Marsden-like identity for $r = 2$, and state it in a form valid for any $r \geq 1$. It is verified symbolically for $r \leq 6$.

1.4.1 The C^2 elements, $\bar{\Sigma}^2_6$

According to Definition 3, we use only 36 elements of $\bar{\Sigma}^2_6$ with indices from the set $\bar{\mathcal{I}}^2$, which corresponds to removing $\triangle_{2223}$ from Σ^2_6. In Definition 4 we list these elements, normalized to ensure partition of unity. The set $\bar{\Sigma}^2_6$ consists of Bernstein polynomials as depicted in Figure 1.2, and 15 other simplex splines with at least one knot at the barycenter of the triangle.

Definition 4 (The functions $\bar{\Sigma}^2_6$)

The functions $\bar{\Sigma}^2_6$ consists of the 21 Bernstein polynomials

$$S^6_{i+1,j+1,k+1,0} := \triangle_{i+1,j+1,k+1;0} = B^6_{ijk}, \qquad (i, j, k) \in \mathcal{I}^2_{\text{Bernstein}}, \tag{1.40}$$

and the additional simplex splines

$$\begin{aligned}
&S^6_{22} := S^6_{4311} := \frac{1}{3}\triangle_{4311}, \quad S^6_{23} := S^6_{3411} := \frac{1}{3}\triangle_{3411}, \quad S^6_{24} := S^6_{1431} := \frac{1}{3}\triangle_{1431}, \\
&S^6_{25} := S^6_{1341} := \frac{1}{3}\triangle_{1341}, \quad S^6_{26} := S^6_{3141} := \frac{1}{3}\triangle_{3141}, \quad S^6_{27} := S^6_{4131} := \frac{1}{3}\triangle_{4131}, \\
&S^6_{28} := S^6_{4221} := \frac{2}{3}\triangle_{4221}, \quad S^6_{30} := S^6_{2421} := \frac{2}{3}\triangle_{2421}, \quad S^6_{32} := S^6_{2241} := \frac{2}{3}\triangle_{2241}, \\
&S^6_{29} := S^6_{3312} := \frac{1}{3}\triangle_{3312}, \quad S^6_{31} := S^6_{1332} := \frac{1}{3}\triangle_{1332}, \quad S^6_{33} := S^6_{3132} := \frac{1}{3}\triangle_{3132}, \\
&S^6_{34} := S^6_{3222} := \frac{2}{3}\triangle_{3222}, \quad S^6_{35} := S^6_{2322} := \frac{2}{3}\triangle_{2322}, \quad S^6_{36} := S^6_{2232} := \frac{2}{3}\triangle_{2232}.
\end{aligned} \tag{1.41}$$

For the numbering of the functions $S_1^6, \dots, S_{21}^6$ see Figures 1.3 and the new figure 1.4 below. In the following we give explicit formulas for the simplex splines in (1.41). On $\mathcal{T}_1$ we have

$$\left(\begin{array}{lll}
22 & \{4,3,1,1\} & \beta_1^6 - 6\beta_2\beta_1^5 + 15\beta_2^2\beta_1^4 \\
23 & \{3,4,1,1\} & -\beta_1^6 + 6\beta_2\beta_1^5 - 15\beta_2^2\beta_1^4 + 20\beta_2^3\beta_1^3 \\
24 & \{1,4,3,1\} & S_{24}^6(1) \\
25 & \{1,3,4,1\} & S_{25}^6(1) \\
26 & \{3,1,4,1\} & -\beta_1^6 + 6\beta_3\beta_1^5 - 15\beta_3^2\beta_1^4 + 20\beta_3^3\beta_1^3 \\
27 & \{4,1,3,1\} & \beta_1^6 - 6\beta_3\beta_1^5 + 15\beta_3^2\beta_1^4 \\
28 & \{4,2,2,1\} & 4\beta_1^6 - 12\beta_2\beta_1^5 - 12\beta_3\beta_1^5 + 60\beta_2\beta_3\beta_1^4 \\
29 & \{3,3,1,2\} & 9\beta_1^6 - 36\beta_2\beta_1^5 + 45\beta_2^2\beta_1^4 \\
30 & \{2,4,2,1\} & 8\beta_1^6 - 36\beta_2\beta_1^5 - 12\beta_3\beta_1^5 + 60\beta_2^2\beta_1^4 + 60\beta_2\beta_3\beta_1^4 - 40\beta_2^3\beta_1^3 - 120\beta_2^2\beta_3\beta_1^3 + 120\beta_2^3\beta_3\beta_1^2 \\
31 & \{1,3,3,2\} & S_{31}^6(1) \\
32 & \{2,2,4,1\} & 8\beta_1^6 - 12\beta_2\beta_1^5 - 36\beta_3\beta_1^5 + 60\beta_3^2\beta_1^4 + 60\beta_2\beta_3\beta_1^4 - 40\beta_3^3\beta_1^3 - 120\beta_2\beta_3^2\beta_1^3 + 120\beta_2\beta_3^3\beta_1^2 \\
33 & \{3,1,3,2\} & 9\beta_1^6 - 36\beta_3\beta_1^5 + 45\beta_3^2\beta_1^4 \\
34 & \{3,2,2,2\} & 36\beta_1^6 - 72\beta_2\beta_1^5 - 72\beta_3\beta_1^5 + 180\beta_2\beta_3\beta_1^4 \\
35 & \{2,3,2,2\} & -72\beta_1^6 + 216\beta_2\beta_1^5 + 108\beta_3\beta_1^5 - 180\beta_2^2\beta_1^4 - 360\beta_2\beta_3\beta_1^4 + 360\beta_2^2\beta_3\beta_1^3 \\
36 & \{2,2,3,2\} & -72\beta_1^6 + 108\beta_2\beta_1^5 + 216\beta_3\beta_1^5 - 180\beta_3^2\beta_1^4 - 360\beta_2\beta_3\beta_1^4 + 360\beta_2\beta_3^2\beta_1^3
\end{array}\right) \tag{1.42}$$

where the first entry denotes the position of the spline with respect to aforementioned ordering, the second the multiplicity of the knots and the third one the explicit expression on $\mathcal{T}_1$, using

$$\begin{aligned}
S_{24}^6(1) &= -10\beta_1^6 + 36\beta_2\beta_1^5 + 24\beta_3\beta_1^5 - 45\beta_2^2\beta_1^4 - 15\beta_3^2\beta_1^4 - 90\beta_2\beta_3\beta_1^4 + 20\beta_2^3\beta_1^3 + 60\beta_2\beta_3^2\beta_1^3 \\
&\quad + 120\beta_2^2\beta_3\beta_1^3 - 90\beta_2^2\beta_3^2\beta_1^2 - 60\beta_2^3\beta_3\beta_1^2 + 60\beta_2^3\beta_3^2\beta_1 \\
S_{25}^6(1) &= -10\beta_1^6 + 24\beta_2\beta_1^5 + 36\beta_3\beta_1^5 - 15\beta_2^2\beta_1^4 - 45\beta_3^2\beta_1^4 - 90\beta_2\beta_3\beta_1^4 + 20\beta_3^3\beta_1^3 \\
&\quad + 120\beta_2\beta_3^2\beta_1^3 + 60\beta_2^2\beta_3\beta_1^3 - 60\beta_2\beta_3^3\beta_1^2 - 90\beta_2^2\beta_3^2\beta_1^2 + 60\beta_2^2\beta_3^3\beta_1 \\
S_{31}^6(1) &= 90\beta_1^6 - 216\beta_2\beta_1^5 - 216\beta_3\beta_1^5 + 135\beta_2^2\beta_1^4 + 135\beta_3^2\beta_1^4 + 540\beta_2\beta_3\beta_1^4 \\
&\quad - 360\beta_2\beta_3^2\beta_1^3 - 360\beta_2^2\beta_3\beta_1^3 + 270\beta_2^2\beta_3^2\beta_1^2
\end{aligned} \tag{1.43}$$

Here, $S_k^6(1)$ is used to indicate the restriction of S_k^6 to the triangle $\mathcal{T}_1$. On $\mathcal{T}_2$ we have, in the same fashion,

$$\left(\begin{array}{lll}
22 & \{4,3,1,1\} & -\beta_2^6 + 6\beta_1\beta_2^5 - 15\beta_1^2\beta_2^4 + 20\beta_1^3\beta_2^3 \\
23 & \{3,4,1,1\} & \beta_2^6 - 6\beta_1\beta_2^5 + 15\beta_1^2\beta_2^4 \\
24 & \{1,4,3,1\} & \beta_2^6 - 6\beta_3\beta_2^5 + 15\beta_3^2\beta_2^4 \\
25 & \{1,3,4,1\} & -\beta_2^6 + 6\beta_3\beta_2^5 - 15\beta_3^2\beta_2^4 + 20\beta_3^3\beta_2^3 \\
26 & \{3,1,4,1\} & S_{26}^6(2) \\
27 & \{4,1,3,1\} & S_{27}^6(2) \\
28 & \{4,2,2,1\} & 8\beta_2^6 - 36\beta_1\beta_2^5 - 12\beta_3\beta_2^5 + 60\beta_1^2\beta_2^4 + 60\beta_1\beta_3\beta_2^4 - 40\beta_1^3\beta_2^3 - 120\beta_1^2\beta_3\beta_2^3 + 120\beta_1^3\beta_3\beta_2^2 \\
29 & \{3,3,1,2\} & 9\beta_2^6 - 36\beta_1\beta_2^5 + 45\beta_1^2\beta_2^4 \\
30 & \{2,4,2,1\} & 4\beta_2^6 - 12\beta_1\beta_2^5 - 12\beta_3\beta_2^5 + 60\beta_1\beta_3\beta_2^4 \\
31 & \{1,3,3,2\} & 9\beta_2^6 - 36\beta_3\beta_2^5 + 45\beta_3^2\beta_2^4 \\
32 & \{2,2,4,1\} & 8\beta_2^6 - 12\beta_1\beta_2^5 - 36\beta_3\beta_2^5 + 60\beta_3^2\beta_2^4 + 60\beta_1\beta_3\beta_2^4 - 40\beta_3^3\beta_2^3 - 120\beta_1\beta_3^2\beta_2^3 + 120\beta_1\beta_3^3\beta_2^2 \\
33 & \{3,1,3,2\} & S_{33}^6(2) \\
34 & \{3,2,2,2\} & -72\beta_2^6 + 216\beta_1\beta_2^5 + 108\beta_3\beta_2^5 - 180\beta_1^2\beta_2^4 - 360\beta_1\beta_3\beta_2^4 + 360\beta_1^2\beta_3\beta_2^3 \\
35 & \{2,3,2,2\} & 36\beta_2^6 - 72\beta_1\beta_2^5 - 72\beta_3\beta_2^5 + 180\beta_1\beta_3\beta_2^4 \\
36 & \{2,2,3,2\} & -72\beta_2^6 + 108\beta_1\beta_2^5 + 216\beta_3\beta_2^5 - 180\beta_3^2\beta_2^4 - 360\beta_1\beta_3\beta_2^4 + 360\beta_1\beta_3^2\beta_2^3
\end{array}\right) \tag{1.44}$$

where

$$\begin{aligned}
S_{26}^6(2) &= -10\beta_2^6 + 24\beta_1\beta_2^5 + 36\beta_3\beta_2^5 - 15\beta_1^2\beta_2^4 - 45\beta_3^2\beta_2^4 - 90\beta_1\beta_3\beta_2^4 + 20\beta_3^3\beta_2^3 \\
&\quad + 120\beta_1\beta_3^2\beta_2^3 + 60\beta_1^2\beta_3\beta_2^3 - 60\beta_1\beta_3^3\beta_2^2 - 90\beta_1^2\beta_3^2\beta_2^2 + 60\beta_1^2\beta_3^3\beta_2 \\
S_{27}^6(2) &= -10\beta_2^6 + 36\beta_1\beta_2^5 + 24\beta_3\beta_2^5 - 45\beta_1^2\beta_2^4 - 15\beta_3^2\beta_2^4 - 90\beta_1\beta_3\beta_2^4 + 20\beta_1^3\beta_2^3 \\
&\quad + 60\beta_1\beta_3^2\beta_2^3 + 120\beta_1^2\beta_3\beta_2^3 - 90\beta_1^2\beta_3^2\beta_2^2 - 60\beta_1^3\beta_3\beta_2^2 + 60\beta_1^3\beta_3^2\beta_2 \\
S_{33}^6(2) &= 90\beta_2^6 - 216\beta_1\beta_2^5 - 216\beta_3\beta_2^5 + 135\beta_1^2\beta_2^4 + 135\beta_3^2\beta_2^4 + 540\beta_1\beta_3\beta_2^4 \\
&\quad - 360\beta_1\beta_3^2\beta_2^3 - 360\beta_1^2\beta_3\beta_2^3 + 270\beta_1^2\beta_3^2\beta_2^2,
\end{aligned} \tag{1.45}$$

and finally on $\mathcal{T}_3$

$$\begin{pmatrix}
22\ \{4,3,1,1\} & S_{22}^6(3) \\
23\ \{3,4,1,1\} & S_{23}^6(3) \\
24\ \{1,4,3,1\} & -\beta_3^6 + 6\beta_2\beta_3^5 - 15\beta_2^2\beta_3^4 + 20\beta_2^3\beta_3^3 \\
25\ \{1,3,4,1\} & \beta_3^6 - 6\beta_2\beta_3^5 + 15\beta_2^2\beta_3^4 \\
26\ \{3,1,4,1\} & \beta_3^6 - 6\beta_1\beta_3^5 + 15\beta_1^2\beta_3^4 \\
27\ \{4,1,3,1\} & -\beta_3^6 + 6\beta_1\beta_3^5 - 15\beta_1^2\beta_3^4 + 20\beta_1^3\beta_3^3 \\
28\ \{4,2,2,1\} & 8\beta_3^6 - 36\beta_1\beta_3^5 - 12\beta_2\beta_3^5 + 60\beta_1^2\beta_3^4 + 60\beta_1\beta_2\beta_3^4 - 40\beta_1^3\beta_3^3 - 120\beta_1^2\beta_2\beta_3^3 + 120\beta_1^3\beta_2\beta_3^2 \\
29\ \{3,3,1,2\} & S_{29}^6(3) \\
30\ \{2,4,2,1\} & 8\beta_3^6 - 12\beta_1\beta_3^5 - 36\beta_2\beta_3^5 + 60\beta_2^2\beta_3^4 + 60\beta_1\beta_2\beta_3^4 - 40\beta_2^3\beta_3^3 - 120\beta_1\beta_2^2\beta_3^3 + 120\beta_1\beta_2^3\beta_3^2 \\
31\ \{1,3,3,2\} & 9\beta_3^6 - 36\beta_2\beta_3^5 + 45\beta_2^2\beta_3^4 \\
32\ \{2,2,4,1\} & 4\beta_3^6 - 12\beta_1\beta_3^5 - 12\beta_2\beta_3^5 + 60\beta_1\beta_2\beta_3^4 \\
33\ \{3,1,3,2\} & 9\beta_3^6 - 36\beta_1\beta_3^5 + 45\beta_1^2\beta_3^4 \\
34\ \{3,2,2,2\} & -72\beta_3^6 + 216\beta_1\beta_3^5 + 108\beta_2\beta_3^5 - 180\beta_1^2\beta_3^4 - 360\beta_1\beta_2\beta_3^4 + 360\beta_1^2\beta_2\beta_3^3 \\
35\ \{2,3,2,2\} & -72\beta_3^6 + 108\beta_1\beta_3^5 + 216\beta_2\beta_3^5 - 180\beta_2^2\beta_3^4 - 360\beta_1\beta_2\beta_3^4 + 360\beta_1\beta_2^2\beta_3^3 \\
36\ \{2,2,3,2\} & 36\beta_3^6 - 72\beta_1\beta_3^5 - 72\beta_2\beta_3^5 + 180\beta_1\beta_2\beta_3^4
\end{pmatrix} \tag{1.46}$$

with

$$\begin{aligned}
S_{22}^6(3) = S_{22}^6|_{\mathcal{T}_3} &= -10\beta_3^6 + 36\beta_1\beta_3^5 + 24\beta_2\beta_3^5 - 45\beta_1^2\beta_3^4 - 15\beta_2^2\beta_3^4 - 90\beta_1\beta_2\beta_3^4 + 20\beta_1^3\beta_3^3 + 60\beta_1\beta_2^2\beta_3^3 \\
&\quad + 120\beta_1^2\beta_2\beta_3^3 - 90\beta_1^2\beta_2^2\beta_3^2 - 60\beta_1^3\beta_2\beta_3^2 + 60\beta_1^3\beta_2^2\beta_3 \\
S_{23}^6(3) = S_{23}^6|_{\mathcal{T}_3} &= -10\beta_3^6 + 24\beta_1\beta_3^5 + 36\beta_2\beta_3^5 - 15\beta_1^2\beta_3^4 - 45\beta_2^2\beta_3^4 - 90\beta_1\beta_2\beta_3^4 + 20\beta_2^3\beta_3^3 \\
&\quad + 120\beta_1\beta_2^2\beta_3^3 + 60\beta_1^2\beta_2\beta_3^3 - 60\beta_1\beta_2^3\beta_3^2 - 90\beta_1^2\beta_2^2\beta_3^2 + 60\beta_1^2\beta_2^3\beta_3 = 60\beta_1^2\beta_2^3\beta_3 + O(\beta_3^2) \\
S_{29}^6(3) = S_{29}^6|_{\mathcal{T}_3} &= 90\beta_3^6 - 216\beta_1\beta_3^5 - 216\beta_2\beta_3^5 + 135\beta_1^2\beta_3^4 + 135\beta_2^2\beta_3^4 + 540\beta_1\beta_2\beta_3^4 \\
&\quad - 360\beta_1\beta_2^2\beta_3^3 - 360\beta_1^2\beta_2\beta_3^3 + 270\beta_1^2\beta_2^2\beta_3^2
\end{aligned} \tag{1.47}$$

The restrictions of the simplex splines can even be written in terms of Bernstein polynomials on the three subtriangles. Here are two examples.

$$\begin{aligned}
S_{22}^6(3) = S_{4311}^6|_{\mathcal{T}_3} = \{&B_{420} - B_{510} + B_{600}, -B_{060} + B_{150} - B_{240} + B_{330}, \\
&- 10B_{006} + 4B_{015} - B_{024} + 6B_{105} - 3B_{114} + B_{123} - 3B_{204} + 2B_{213} - B_{222} + B_{303} - B_{312} + B_{321}\}
\end{aligned}$$

$$\begin{aligned}
S_{23}^6(3) = S_{3411}^6|_{\mathcal{T}_3} = \{&-B_{600} + B_{510} - B_{420} + B_{330}, B_{240} - B_{150} + B_{060}, \\
&- 10B_{006} + 4B_{105} - B_{204} + 6B_{015} - 3B_{114} + B_{213} - 3B_{024} + 2B_{123} - B_{222} + B_{033} - B_{132} + B_{231}\}
\end{aligned}$$

Since $\max\{i,j,k\} + \ell \le 5$ for all elements $S_{ijk\ell}^6 \in \bar{\Sigma}_6^2$ with $\ell > 0$, it follows from the local smoothness property of simplex splines that $\bar{\Sigma}_6^2 \subseteq \bar{\mathbb{S}}_6^2(\triangle)$. Moreover, the number of elements in $\bar{\Sigma}_6^2$ is equal to the dimension 36 of $\bar{\mathbb{S}}_6^2(\triangle)$. Therefore, the following proposition implies that $\bar{\Sigma}_6^2$ is a basis for $\bar{\mathbb{S}}_6^2(\triangle)$.

The following result is, of course, a special case of Proposition 3, but we provide a direct proof based on the explicit expressions in the appendix

Proposition 4 *The functions in $\bar{\Sigma}_6^2$ are linearly independent on $\triangle$.*

Now we are in position to formulate and prove the announced Marsden identity.

Theorem 2 (Barycentric Marsden-like identity for $d = 6$)
For $\boldsymbol{u} := [u_1, u_2, u_3]^T \in \mathbb{R}^3, \boldsymbol{\beta} := [\beta_1, \beta_2, \beta_3]^T \in \mathbb{R}^3$, with $\beta_i \ge 0$, $i = 1,2,3$ and $\beta_1 + \beta_2 + \beta_3 = 1$ we have

$$(\boldsymbol{u}^T\boldsymbol{\beta})^6 = \sum_{(i,j,k\ell)\in\bar{\mathcal{I}}^2} \rho_{ijk\ell}(\boldsymbol{u}) S_{ijk\ell}^6(\boldsymbol{\beta}),$$

where, with $\bar{u}_{m,n} := (u_m + u_n)/2$ for $m,n = 1,2,3$, and $\bar{u}_{123} := (u_1 + u_2 + u_3)/3$,

$$\rho_{i+1,j+1,k+1,0}(\boldsymbol{u}) = u_1^i u_2^j u_3^k, \qquad (i,j,k) \in \mathcal{I}_{Bernstein}^2, \tag{1.48}$$

and

$$
\begin{array}{lll}
\rho_{4311}(\boldsymbol{u}) = u_1^3u_2^2u_3, & \rho_{4131}(\boldsymbol{u}) = u_1^3u_2u_3^2, & \rho_{3411}(\boldsymbol{u}) = u_1^2u_2^3u_3,\\
\rho_{3141}(\boldsymbol{u}) = u_1^2u_2u_3^3, & \rho_{1431}(\boldsymbol{u}) = u_1u_2^3u_3^2, & \rho_{1341}(\boldsymbol{u}) = u_1u_2^2u_3^3,\\
\rho_{4221}(\boldsymbol{u}) = u_1^3u_2u_3\bar{u}_{2,3}, & \rho_{2421}(\boldsymbol{u}) = u_1u_2^3u_3\bar{u}_{1,3}, & \rho_{2241}(\boldsymbol{u}) = u_1u_2u_3^3\bar{u}_{1,2}\\
\rho_{3312}(\boldsymbol{u}) = u_1^2u_2^2u_3\bar{u}_{123}, & \rho_{3132}(\boldsymbol{u}) = u_1^2u_2u_3^2\bar{u}_{123}, & \rho_{1332}(\boldsymbol{u}) = u_1u_2^2u_3^2\bar{u}_{123},\\
\rho_{3222}(\boldsymbol{u}) = u_1^2u_2u_3\bar{u}_{2,3}\bar{u}_{123}, & \rho_{2322}(\boldsymbol{u}) = u_1u_2^2u_3\bar{u}_{1,3}\bar{u}_{123}, & \rho_{2232}(\boldsymbol{u}) = u_1u_2u_3^2\bar{u}_{1,2}\bar{u}_{123}.
\end{array}
\tag{1.49}
$$

Proof The barycentric Marsden-like identity follows from the barycentric form (1.3) of the Marsden identity for Bernstein polynomials by expressing the removed Bernstein polynomials in terms of the elements in $\bar{\Sigma}_6^2$. Here are some details. For $(i,j,k) \in \mathcal{I}^2_{\text{removed}}$, where

$$
\mathcal{I}^2_{\text{removed}} := \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1),(2,2,2)\},
$$

we insert knots at the barycenter using (1.18), (1.41), and find

$$
\begin{aligned}
B^6_{1,2,3} &:= \triangle_{2340} = \big(\triangle_{1341} + \triangle_{2241} + \triangle_{2331}\big)/3\\
&= \big(\triangle_{1341} + \triangle_{2241}\big)/3 + \big(\triangle_{1332} + \triangle_{2232} + \triangle_{2322}\big)/9,\\
B^6_{1,3,2} &= \big(\triangle_{1431} + \triangle_{2421}\big)/3 + \big(\triangle_{1332} + \triangle_{2232} + \triangle_{2322}\big)/9,\\
B^6_{2,1,3} &= \big(\triangle_{2241} + \triangle_{3141}\big)/3 + \big(\triangle_{3132} + \triangle_{2232} + \triangle_{3222}\big)/9,\\
B^6_{2,3,1} &= \big(\triangle_{2421} + \triangle_{3411}\big)/3 + \big(\triangle_{3312} + \triangle_{2322} + \triangle_{3222}\big)/9,\\
B^6_{3,1,2} &= \big(\triangle_{4131} + \triangle_{4221}\big)/3 + \big(\triangle_{3132} + \triangle_{2232} + \triangle_{3222}\big)/9,\\
B^6_{3,2,1} &= \big(\triangle_{4221} + \triangle_{4311}\big)/3 + \big(\triangle_{3312} + \triangle_{2322} + \triangle_{3222}\big)/9,\\
B^6_{2,2,2} &= \big(\triangle_{1332} + \triangle_{3132} + \triangle_{3312} + 2\triangle_{2232} + 2\triangle_{2322} + 2\triangle_{3222}\big)/9.
\end{aligned}
\tag{1.50}
$$

By (1.3)

$$
(\boldsymbol{u}^T\boldsymbol{\beta})^6 = \sum_{(i,j,k)\in\mathcal{I}^2_{\text{Bernstein}}} u_1^iu_2^ju_3^kB^6_{ijk}(\boldsymbol{\beta}) + \sum_{(i,j,k)\in\mathcal{I}^2_{\text{removed}}} u_1^iu_2^ju_3^kB^6_{ijk}(\boldsymbol{\beta}).
$$

For $(i,j,k) \in \mathcal{I}^2_{\text{Bernstein}}$ we have $B^6_{ijk}(\boldsymbol{\beta}) = S^6_{i+1,j+1,k+1,0}(\boldsymbol{\beta})$ and hence $\rho_{[i+1,j+1,k+1,0]} = u_1^iu_2^ju_3^k$. In the second sum we insert the expressions in (1.50) for B^6_{ijk}, and collect terms for each $\triangle_{ijk\ell}$ to obtain (1.49). We show this for three typical cases.

$$
\begin{aligned}
\sum_{(i,j,k)\in\mathcal{I}^2_{\text{removed}}} u_1^iu_2^ju_3^kB^6_{ijk}(\boldsymbol{\beta}) &= u_1u_2^2u_3^3\triangle_{1431}(\boldsymbol{\beta})/3 + \big(u_1u_2^2u_3^3 + u_1u_2^3u_3^2 + u_1^2u_2^2u_3^2\big)\triangle_{1332}(\boldsymbol{\beta})/9\\
&\quad + \big(u_1u_2^2u_3^3 + u_1u_2^3u_3^2 + u_1^2u_2u_3^3 + u_1^3u_2u_3^2 + 2u_1^2u_2^2u_3^2\big)\triangle_{2232}(\boldsymbol{\beta})/9 + \cdots\\
&= u_1u_2^2u_3^3\triangle_{1431}(\boldsymbol{\beta})/3 + u_1u_2^2u_3^2\bar{u}_{123}\triangle_{1332}(\boldsymbol{\beta})/3 + 2u_1u_2u_3^2\bar{u}_{1,2}\bar{u}_{123}\triangle_{2232}(\boldsymbol{\beta})/3 + \cdots,\\
&= u_1u_2^2u_3^3S^6_{1431}(\boldsymbol{\beta}) + u_1u_2^2u_3^2\bar{u}_{123}S^6_{1332}(\boldsymbol{\beta}) + u_1u_2u_3^2\bar{u}_{1,2}\bar{u}_{123}S^6_{2232}(\boldsymbol{\beta}) + \cdots,
\end{aligned}
$$

and (1.49) follows.
■

Remark 3 Since $\rho_{ijk\ell}(1,1,1) = 1$ for any $ijk\ell$ it follows that

$$
\sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} S^6_{ijk\ell}(\boldsymbol{\beta}) = 1.
$$

Corollary 1 (Domain points for $d = 6$)
The domain points $\boldsymbol{p}^*_{ijk\ell}$ in barycentric form, defined as the coefficients in the expansion

$$
\boldsymbol{\beta} = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} \boldsymbol{p}^*_{ijk\ell}S^6_{ijk\ell}(\boldsymbol{\beta}).
$$

are given by

$$\boldsymbol{p}^*_{i+1,j+1,k+1,0} = (i,j,k)/6, \qquad (i,j,k) \in \mathcal{I}^2_{Bernstein},$$

and moreover

$$\begin{array}{lll}
\boldsymbol{p}^*_{4311} = (3,2,1)/6, & \boldsymbol{p}^*_{4131} = (3,1,2)/6, & \boldsymbol{p}^*_{3411} = (2,3,1)/6, \\
\boldsymbol{p}^*_{3141} = (2,1,3)/6, & \boldsymbol{p}^*_{1431} = (1,3,2)/6, & \boldsymbol{p}^*_{1341} = (1,2,3)/6, \\
\boldsymbol{p}^*_{3312} = (7,7,4)/18, & \boldsymbol{p}^*_{3132} = (7,4,7)/18, & \boldsymbol{p}^*_{1332} = (4,7,7)/18, \\
\boldsymbol{p}^*_{4221} = (2,1,1)/4, & \boldsymbol{p}^*_{2421} = (1,2,1)/4, & \boldsymbol{p}^*_{2241} = (1,1,2)/4, \\
\boldsymbol{p}^*_{3222} = (14,11,11)/36, & \boldsymbol{p}^*_{2322} = (11,14,11)/36, & \boldsymbol{p}^*_{2232} = (11,11,14)/36.
\end{array} \tag{1.51}$$

Proof By the Marsden-like identity we have

$$\beta_m = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} \frac{\partial}{\partial u_m} \rho_{ijk\ell}(1,1,1) S^6_{ijk\ell}(\boldsymbol{\beta}), \qquad m = 1,2,3,$$

and (1.51) follows after a straightforward calculation.
■

The indices i,j,k and i,j,k,ℓ for the domain points are shown in Figure 1.3.

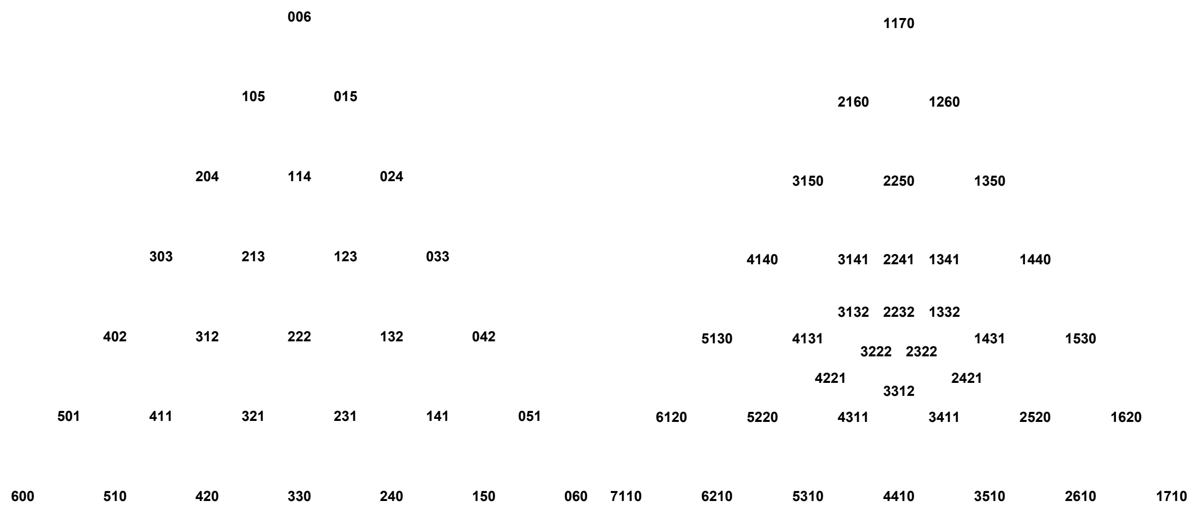


Fig. 1.3 Bernstein domain points on the left, Simplex splines domain points on the right.

Next, we define, as usually, the ∞-norm condition number of the $\bar{\Sigma}^r_{3r}$ bases for $\bar{\mathbb{S}}^r_{3r}(\triangle)$ by

$$\kappa_{d,\infty}(\mathcal{T}) := \max_{\boldsymbol{c}\neq\boldsymbol{0}} \frac{\|\boldsymbol{b}^T\boldsymbol{c}\|_{L_\infty(\mathcal{T})}}{\|\boldsymbol{c}\|_\infty} \max_{\boldsymbol{c}\neq\boldsymbol{0}} \frac{\|\boldsymbol{c}\|_\infty}{\|\boldsymbol{b}^T\boldsymbol{c}\|_{L_\infty(\mathcal{T})}},$$

where $d = 3r$ and $\boldsymbol{b}^T\boldsymbol{c} := \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^r} c_{ijk\ell} S^d_{ijk\ell} \in \bar{\mathbb{S}}^r_d(\triangle)$. This number turns out to be bounded by a moderate number *independently* of the shape of the basis triangle $\mathcal{T}$.

Proposition 5 (Stability) *For any triangle $\mathcal{T}$ we have $\kappa_{6,\infty}(\mathcal{T}) < 1350$.*

Proof Since the $S^6_{ijk\ell}$, $(i,j,k,\ell) \in \bar{\mathcal{I}}^2$ form a nonnegative partition of unity it follows that for any $\boldsymbol{x} \in \triangle$

$$|\boldsymbol{b}(\boldsymbol{x})^T\boldsymbol{c}| \le \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} |c_{ijk\ell}|\,|S^6_{ijk\ell}(\boldsymbol{x})| \le \|\boldsymbol{c}\|_\infty \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} S^6_{ijk\ell}(\boldsymbol{x}) = \|\boldsymbol{c}\|_\infty$$

with an equality if all the $c_{ijk\ell} = 1$, so that $\max_{\boldsymbol{c}\neq\boldsymbol{0}} \|\boldsymbol{b}^T\boldsymbol{c}\|_{L_\infty(\mathcal{T})}/\|\boldsymbol{c}\|_\infty = 1$.

To bound the second part of $\kappa_{6,\infty}(\mathcal{T})$, we consider a spline $s \in \bar{\mathbb{S}}^2_6$ interpolating given data at the 36 domain points. Using the ordering of domain points $\boldsymbol{p}^*_m$ and basis functions S^6_n, $m,n = 1,\ldots,36$, shown in Figure 1.4 we obtain a linear system

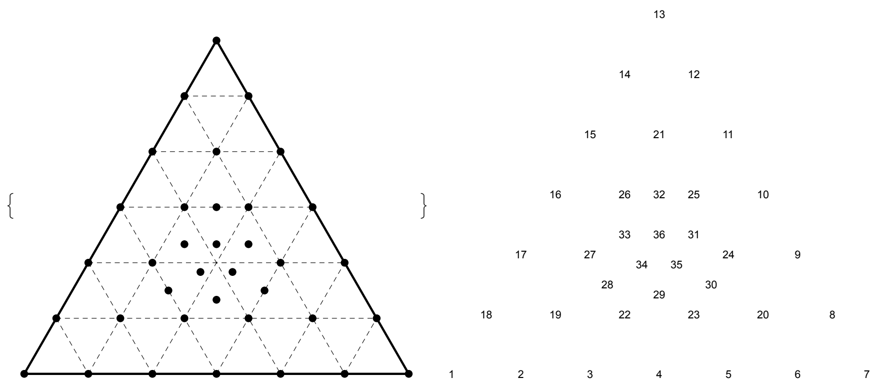


Fig. 1.4 The position, left, and ordering of the domainpoints and corresponding basis functions in Figure 1.3, right.

$$\boldsymbol{A}\boldsymbol{c} = \begin{bmatrix} S_1^6(\boldsymbol{p}_1^*) & \cdots & S_{36}^6(\boldsymbol{p}_1^*) \\ \vdots & \vdots & \\ S_1^6(\boldsymbol{p}_{36}^*) & \cdots & S_{36}^6(\boldsymbol{p}_{36}^*) \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_{36} \end{bmatrix} = \begin{bmatrix} s(\boldsymbol{p}_1^*) \\ \vdots \\ s(\boldsymbol{p}_{36}^*) \end{bmatrix} = \boldsymbol{s}^*$$

for the coeffcients $\boldsymbol{c} := [c_1, \ldots, c_{36}]^T$ of $s := \sum_{n=1}^{36} c_n S_n^6$. Here $S_1^6, \ldots, S_{21}^6$ are Bernstein polynomials, and $S_n^6(\boldsymbol{p}_m^*) = 0$ for $m = 1, \ldots, 18$ and $n = 19, \ldots, 36$, by continuity properties of simplex splines. Thus $\boldsymbol{A}$ has the lower triangular block form $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{A}_2 & \boldsymbol{A}_3 \end{bmatrix}$, where $\boldsymbol{A}_m, \in \mathbb{R}^{18\times 18}$, for $m = 1, 2, 3$. Using symbolic computation it follows that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are nonsingular. Thus $\boldsymbol{A}$ is nonsingular with inverse $\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 \end{bmatrix}$, and $\boldsymbol{B}_1 = \boldsymbol{A}_1^{-1}$, $\boldsymbol{B}_3 = \boldsymbol{A}_3^{-1}$, $\boldsymbol{B}_2 = -\boldsymbol{B}_3\boldsymbol{A}_2\boldsymbol{B}_1$. We compute

$$\|\boldsymbol{A}^{-1}\|_\infty = \frac{12\,209\,201\,545\,461}{9\,044\,604\,800} < 1350 - \frac{1}{9},$$

and the proposition follows.
■

Finally, we consider next the problem of obtaining C^2–continuity across an edge between two adjacent triangles in a global triangulation, making use of the local properties of our simplex spline basis.

We begin with a technical observation.

Lemma 4 *On* $\mathcal{T}_3$ *i.e. for* $0 \le \beta_3 \le \beta_1 \le 1$ *and* $0 \le \beta_3 \le \beta_2 \le 1$

$$\begin{aligned} S_{i+1,j+1,k+1,0}^6 &= B_{ijk}^6, \ \{(i,j,k) : i+j+k = 6,\ i \ge 4 \text{ or } j \ge 4\}, \\ S_{4311}^6|_{\mathcal{T}_3} &= B_{321}^6 - B_{312}^6 - B_{222}^6 + O(\beta_3^3), \\ S_{3411}^6|_{\mathcal{T}_3} &= B_{231}^6 - B_{222}^6 - B_{132}^6 + O(\beta_3^3), \\ S_{4221}^6|_{\mathcal{T}_3} &= 2B_{312}^6 + O(\beta_3^3),\ S_{3312}^6|_{\mathcal{T}_3} = 3B_{222}^6 + O(\beta_3^3),\ S_{2421}^6|_{\mathcal{T}_3} = 2B_{132}^6 + O(\beta_3^3) \\ S_{3222}^6|_{\mathcal{T}_3} &= 6B_{213}^6 + O(\beta_3^4)\ S_{2322}^6|_{\mathcal{T}_3} = 6B_{123}^6 + O(\beta_3^4). \end{aligned} \tag{1.52}$$

while $S_{ijk\ell}^6|_{\mathcal{T}_3} = O(\beta_3^3)$ *for the remaining splines.*

Proof The set $S_{i+1,j+1,k+1,0}^6$ with $\{(i,j,k) : i+j+k = 6,\ i \ge 4$ or $j \ge 4\}$ are $2 \times 6 = 12$ classical Bernstein polynomials on the triangle $\mathcal{T}$, located on the lower left and right corners in Figure 1.3. Using the explicit forms (1.46) and (1.47), (1.52) follows by inspection.
■

For

$$s = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^2} c_{ijk\ell} S_{ijk\ell}^6$$

let

$$\begin{aligned} \boldsymbol{c}_0 &= [c_{7110}, c_{6210}, c_{5310}, c_{4410}, c_{3510}, c_{2610}, c_{1710}]^T \in \mathbb{R}^7, \\ \boldsymbol{c}_1 &= [c_{6120}, c_{5220}, c_{4311}, c_{3411}, c_{2520}, c_{1620}]^T \in \mathbb{R}^6, \\ \boldsymbol{c}_2 &= [c_{5130}, c_{4221}, c_{3312}, c_{2421}, c_{1530}]^T \in \mathbb{R}^5, \end{aligned} \tag{1.53}$$

be the coefficients involved in obtaining C^2 continuity across an edge between two triangles, see Figure 1.3.

Proposition 6 *Let*

$$s = \sum_{(i,j,k,\ell)\in\bar{I}^2} c_{ijk\ell} S^6_{ijk\ell} \qquad or \qquad \tilde{s} = \sum_{(i,j,k,\ell)\in\bar{I}^2} \tilde{c}_{ijk\ell} \tilde{S}^6_{ijk\ell},$$

respectively, be defined on the triangle $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ *(resp.* $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3 \rangle$*). We suppose that* $\tilde{\boldsymbol{p}}_3 = \lambda_1 \boldsymbol{p}_1 + \lambda_2 \boldsymbol{p}_2 + \lambda_3 \boldsymbol{p}_3$ *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *The function* $s_+ = \begin{cases} s & on\ \mathcal{T} \\ \tilde{s} & on\ \tilde{\mathcal{T}} \end{cases}$ *is* C^r *with* $r \le 2$ *if and only if*

$$\begin{bmatrix} \tilde{\boldsymbol{c}}_0 \\ \vdots \\ \tilde{\boldsymbol{c}}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_{00} & \dots & \boldsymbol{C}_{0r} \\ \vdots & & \vdots \\ \boldsymbol{C}_{r0} & \dots & \boldsymbol{C}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_0 \\ \vdots \\ \boldsymbol{c}_r \end{bmatrix}, \tag{1.54}$$

where the $\tilde{\boldsymbol{c}}_0, \tilde{\boldsymbol{c}}_1, \tilde{\boldsymbol{c}}_2$ *are the coefficients of* $\tilde{s}$ *corresponding to* (1.53)*, and the matrices* $\boldsymbol{C}_{mn}$ *are defined by*

$$\boldsymbol{C}_{00} = \boldsymbol{I} \in \mathbb{R}^{7\times 7}, \quad \boldsymbol{C}_{01} = \boldsymbol{0} \in \mathbb{R}^{7\times 6}, \quad \boldsymbol{C}_{02} = \boldsymbol{0} \in \mathbb{R}^{7\times 5}, \tag{1.55}$$

$$\boldsymbol{C}_{10} = \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_1 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_1 & \lambda_2 \end{bmatrix} \in \mathbb{R}^{6\times 7}, \quad \boldsymbol{C}_{11} = \lambda_3 \boldsymbol{I} \in \mathbb{R}^{6\times 6}, \quad \boldsymbol{C}_{12} = \boldsymbol{0} \in \mathbb{R}^{6\times 5}, \tag{1.56}$$

$$\begin{aligned} \boldsymbol{C}_{20} &= \begin{bmatrix} \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2/2 & \lambda_1(1/2+\lambda_2) & \lambda_2(1+\lambda_2)/2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1(1+\lambda_1)/3 & (\lambda_1+\lambda_2+2\lambda_1\lambda_2)/3 & \lambda_2(1+\lambda_2)/3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1(1+\lambda_1)/2 & \lambda_2(1/2+\lambda_1) & \lambda_2^2/2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 \end{bmatrix} \in \mathbb{R}^{5\times 7}, \\ \boldsymbol{C}_{21} &= \begin{bmatrix} 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & \lambda_3(1/2-\lambda_3/2+\lambda_2) & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(1-\lambda_3+2\lambda_1)/3 & \lambda_3(1-\lambda_3+2\lambda_2)/3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3(1/2-\lambda_3/2+\lambda_1) & \lambda_2\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 \end{bmatrix} \in \mathbb{R}^{5\times 6}, \\ \boldsymbol{C}_{22} &= \lambda_3^2 \boldsymbol{I} \in \mathbb{R}^{5\times 5}. \end{aligned} \tag{1.57}$$

Proof We begin by the C^r–continuity for the Bézier surfaces using the Bernstein basis. Let $\sigma = \sum_{\nu+\mu+\kappa=6} \gamma_{\nu\mu\kappa} B^6_{\nu\mu\kappa} \in \mathbb{P}_6$ be defined on the triangle $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ (respectively $\tilde{\sigma} = \sum_{\nu+\mu+\kappa=6} \tilde{\gamma}_{\nu\mu\kappa} \tilde{B}^6_{\nu\mu\kappa} \in \mathbb{P}_6$ on $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3 \rangle$) where $\tilde{B}^6_{\nu\mu\kappa}$ are the Bernstein polynomials with barycentric variables $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$.

We recall, see [17, Theorem 2.28], that the function $\sigma_+ = \begin{cases} \sigma & \text{on } \mathcal{T} \\ \tilde{\sigma} & \text{on } \tilde{\mathcal{T}} \end{cases}$ is C^r if and only if

$$[\tilde{\boldsymbol{\gamma}}_m]_{m=0,\dots,r} = [\boldsymbol{\Gamma}_{mn}]_{m,n=0,\dots,r} [\boldsymbol{\gamma}_n]_{n=0,\dots,r}, \tag{1.58}$$

where

$$\begin{aligned} \boldsymbol{\gamma}_0 &= [\gamma_{600}, \gamma_{510}, \gamma_{420}, \gamma_{330}, \gamma_{240}, \gamma_{150}, \gamma_{160}]^T, \\ \boldsymbol{\gamma}_1 &= [\gamma_{501}, \gamma_{411}, \gamma_{321}, \gamma_{231}, \gamma_{141}, \gamma_{051}]^T, \\ \boldsymbol{\gamma}_2 &= [\gamma_{402}, \gamma_{312}, \gamma_{222}, \gamma_{132}, \gamma_{042}]^T, \end{aligned}$$

similarly for the $\tilde{\boldsymbol{\gamma}}_m$ and the matrices are defined by

$$\boldsymbol{\Gamma}_{00} = \boldsymbol{I} \in \mathbb{R}^{7\times 7}, \quad \boldsymbol{\Gamma}_{01} = \boldsymbol{0} \in \mathbb{R}^{7\times 6}, \quad \boldsymbol{\Gamma}_{02} = \boldsymbol{0} \in \mathbb{R}^{7\times 5}, \tag{1.59}$$

$$\mathbf{\Gamma}_{10} = \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_1 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_1 & \lambda_2 \end{bmatrix} \in \mathbb{R}^{6\times 7}, \quad \mathbf{\Gamma}_{11} = \lambda_3 \boldsymbol{I} \in \mathbb{R}^{6\times 6}, \quad \mathbf{\Gamma}_{12} = \mathbf{0} \in \mathbb{R}^{6\times 5}, \tag{1.60}$$

$$\begin{aligned} \mathbf{\Gamma}_{20} &= \begin{bmatrix} \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 & 0 & \dots & 0 \\ 0 & \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 \end{bmatrix} \in \mathbb{R}^{5\times 7}, \\ \mathbf{\Gamma}_{21} &= \begin{bmatrix} 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 & 0 & \dots & 0 \\ 0 & 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 \end{bmatrix} \in \mathbb{R}^{5\times 6}, \quad \mathbf{\Gamma}_{22} = \lambda_3^2 \boldsymbol{I} \in \mathbb{R}^{5\times 5}. \end{aligned} \tag{1.61}$$

See also Figure 1.3.

Consider now the function $s_+ = \begin{cases} s & \text{on } \mathcal{T} \\ \tilde{s} & \text{on } \tilde{\mathcal{T}} \end{cases}$ on $\mathcal{T} \cup \tilde{\mathcal{T}}$. To study the C^r–continuity through the edge $\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \rangle$, it is sufficient to consider s on $\mathcal{T}_3$ and $\tilde{s}$ on $\tilde{\mathcal{T}}_3$. On $\mathcal{T}_3$ (resp. $\tilde{\mathcal{T}}_3$), any $S^6_{ijk\ell}$ of the basis (resp. $\tilde{S}^6_{ijk\ell}$) is a polynomial of degree at most 6, $S^6_{ijk\ell} = \sum_{\nu+\mu+\kappa=6} w^{ijk\ell}_{\nu\mu\kappa} B^6_{\nu\mu\kappa}$ (resp. $\tilde{S}^6_{ijk\ell} = \sum_{\nu+\mu+\kappa=6} w^{ijk\ell}_{\nu\mu\kappa} \tilde{B}^6_{\nu\mu\kappa}$). So that $s|_{\mathcal{T}_3}$ and $\tilde{s}|_{\tilde{\mathcal{T}}_3}$ can also be written in the Bernstein bases

$$\begin{aligned} s|_{\mathcal{T}_3} &= \sum_{(i,j,k,\ell)\in \bar{\mathcal{I}}^2} c_{ijk\ell} S^6_{ijk\ell}|_{\mathcal{T}_3} = \sum_{\nu+\mu+\kappa=6} \gamma_{\nu\mu\kappa} B^6_{\nu\mu\kappa} \text{ with } \gamma_{\nu\mu\kappa} = \sum_{(i,j,k,\ell)\in \bar{\mathcal{I}}^2} c_{ijk\ell} w^{ijk\ell}_{\nu\mu\kappa} \\ \tilde{s}|_{\tilde{\mathcal{T}}_3} &= \sum_{(i,j,k,\ell)\in \bar{\mathcal{I}}^2} \tilde{c}_{ijk\ell} \tilde{S}^6_{ijk\ell}|_{\mathcal{T}_3} = \sum_{\nu+\mu+\kappa=6} \tilde{\gamma}_{\nu\mu\kappa} \tilde{B}^6_{\nu\mu\kappa} \text{ with } \tilde{\gamma}_{\nu\mu\kappa} = \sum_{(i,j,k,\ell)\in \bar{\mathcal{I}}^2} \tilde{c}_{ijk\ell} \tilde{w}^{ijk\ell}_{\nu\mu\kappa}. \end{aligned}$$

From (1.52), we deduce the components $\tilde{w}^{ijk\ell}_{\nu\mu\kappa}$ for $\kappa = 0, 1, 2$ and we put forward the corresponding components

$$\begin{aligned} s|_{\mathcal{T}_3} = {} & c_{7110} B^6_{600} + c_{6210} B^6_{510} + c_{5310} B^6_{420} + c_{4410} B^6_{330} + c_{3510} B^6_{240} + c_{2610} B^6_{150} + c_{1710} B^6_{060} \\ & + c_{6120} B^6_{501} + c_{5220} B^6_{411} + c_{4311} B^6_{321} + c_{3411} B^6_{231} + c_{2520} B^6_{141} + c_{1620} B^6_{051} \\ & + c_{5130} B^6_{402} + (2c_{4221} - c_{4311}) B^6_{312} + (3c_{3312} - c_{4311} - c_{3411}) B^6_{222} \\ & + (2c_{2421} - c_{3411}) B^6_{132} + c_{1530} B^6_{042} \\ & + O(\beta_3^3) \end{aligned}$$

and a similar expression for $\tilde{s}|_{\tilde{\mathcal{T}}_3}$

The conditions for the regularity C^0 of s_+, (1.55), is a consequence of (1.59) and similarly for C^1 with also (1.56) coming from (1.60). To obtain C^2, we add the conditions (1.61). They can be rewritten

$$\begin{aligned} \tilde{c}_{5130} &= c_{7110}\lambda_1^2 + 2c_{6210}\lambda_1\lambda_2 + c_{5310}\lambda_2^2 + 2c_{5220}\lambda_2\lambda_3 + c_{5130}\lambda_3^2 + 2c_{6120}\lambda_1\lambda_3, \\ 2\tilde{c}_{4221} - \tilde{c}_{4311} &= c_{6210}\lambda_1^2 + 2c_{5310}\lambda_1\lambda_2 + c_{4410}\lambda_2^2 + 2c_{4311}\lambda_2\lambda_3 \\ &\quad + (2c_{4221} - c_{4311})\lambda_3^2 + 2c_{5220}\lambda_1\lambda_3, \\ 3\tilde{c}_{3312} - \tilde{c}_{4311} - \tilde{c}_{3411} &= c_{5310}\lambda_1^2 + 2c_{4410}\lambda_1\lambda_2 + c_{3510}\lambda_2^2 + 2c_{3411}\lambda_2\lambda_3 \\ &\quad + (3c_{3312} - c_{4311} - c_{3411})\lambda_3^2 + 2c_{4311}\lambda_1\lambda_3 \\ 2\tilde{c}_{2421} - \tilde{c}_{3411} &= c_{4410}\lambda_1^2 + 2c_{3510}\lambda_1\lambda_2 + c_{2610}\lambda_2^2 + 2c_{2520}\lambda_2\lambda_3 \\ &\quad + (2c_{2421} - c_{3411})\lambda_3^2 + 2c_{3411}\lambda_1\lambda_3, \\ \tilde{c}_{1530} &= c_{3510}\lambda_1^2 + 2c_{2610}\lambda_1\lambda_2 + c_{1710}\lambda_2^2 + c_{1620}\lambda_2\lambda_3 + c_{1530}\lambda_3^2 + 2c_{2520}\lambda_1\lambda_3. \end{aligned}$$ □

We already have $\tilde{c}_{4311}$ and $\tilde{c}_{3411}$ in (1.56) so that we deduce that the last components have to satisfy (1.57).
■

Several examples have been considered for scattered data on the CT-split, see for example [11, 21]. Here, we consider a surface on two triangles, see Figure 1.5. With the 18 conditions from Proposition 6, we obtain a C^2 surface on the two triangles.

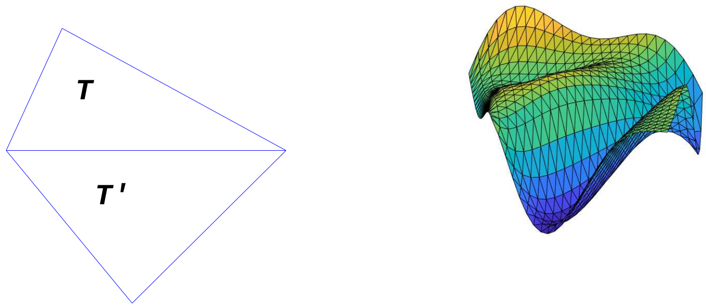


Fig. 1.5 A C^2 surface on two triangles

1.4.2 The C^3 elements, $\bar{\Sigma}_9^3 = \Sigma_9^3$

In this section we repeat the process of the preceding one for $r = 3$. The methods are the same, but the expression become more lengthy. For that reason, we essentially list the results.

The partition of unity basis $\bar{\Sigma}_9^3$ is constructed from the 75 elements $\triangle_{ijk\ell\ell}$ defined in Example 3 as follows,

$$S^9_{ijk\ell} = c_{ijk\ell}\triangle_{ijk\ell}, \quad c_{ijk\ell} = \begin{cases} 1 \text{ if } \ell = 0, \\ 1/3 \text{ if } (\ell > 0 \text{ and } \min(i,j,k) = 1) \text{ or } \ell = 4, \\ 2/3 \text{ if } 0 < \ell < 4 \text{ and } \min(i,j,k) > 1, \end{cases} \qquad \{i,j,k,\ell\} \in \bar{\mathcal{I}}^3. \tag{1.62}$$

Theorem 3 (Barycentric Marsden-like identity for $d = 9$)
For $\boldsymbol{u} := [u_1, u_2, u_3]^T \in \mathbb{R}^3$, $\boldsymbol{\beta} := [\beta_1, \beta_2, \beta_3]^T \in \mathbb{R}^3$, with $\beta_i \geq 0$, $i = 1,2,3$ and $\beta_1 + \beta_2 + \beta_3 = 1$ we have

$$(\boldsymbol{u}^T \boldsymbol{\beta})^9 = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^3} \rho_{ijk\ell}(\boldsymbol{u}) S^9_{ijk\ell}(\boldsymbol{\beta}), \tag{1.63}$$

where

$$\rho_{i+1,j+1,k+1,\ell}(\boldsymbol{u}) = \begin{cases} u_1^i u_2^j u_3^k, & (i,j,k) \in \mathcal{I}^3_{Bernstein}, \,\&\, \ell = 0, \\ u_1^2 u_2^2 u_3^2 \bar{u}_{123}^3, & \ell = 4, \\ u_1^{\max(i,1)} u_2^{\max(j,1)} u_3^{\max(k,1)} \mu_{ijk\ell} \bar{u}_{123}^{\ell-1}, & \text{otherwise} . \end{cases} \tag{1.64}$$

Here $\bar{u}_{123} := (u_1 + u_2 + u_3)/3$, and

$$\mu_{ijk\ell} := \begin{cases} 1, & \text{if } 1,2 \text{ is not among } (i,j,k), \\ (u_r + u_s)/2 & \text{if } 1,2 \text{ is at position } r,s \text{ in } (i,j,k). \end{cases}$$

Proof As for $r = 2$ the barycentric Marsden-like identity follows from the barycentric form (1.3) of the Marsden identity for Bernstein polynomials by expressing the 19 removed Bernstein polynomials B^9_{ijk} in terms of the elements in $\bar{\Sigma}_9^3$. For $(i,j,k) \in \mathcal{I}^3_{\text{removed}}$

$$\begin{aligned}
B^9_{1,3,5} &= \big(\triangle_{1461} + \triangle_{2361}\big)/3 + \big(\triangle_{1452} + \triangle_{2352}\big)/9 + \big(\triangle_{1443} + \triangle_{2343} + \triangle_{2433}\big)/27 \\
B^9_{1,5,3} &= \big(\triangle_{1641} + \triangle_{2631}\big)/3 + \big(\triangle_{1542} + \triangle_{2532}\big)/9 + \big(\triangle_{1443} + \triangle_{2433} + \triangle_{2343}\big)/27 \\
B^9_{5,1,3} &= \big(\triangle_{6141} + \triangle_{6231}\big)/3 + \big(\triangle_{5142} + \triangle_{5232}\big)/9 + \big(\triangle_{4143} + \triangle_{4233} + \triangle_{3243}\big)/27 \\
B^9_{5,3,1} &= \big(\triangle_{6411} + \triangle_{6321}\big)/3 + \big(\triangle_{5412} + \triangle_{5322}\big)/9 + \big(\triangle_{4413} + \triangle_{4323} + \triangle_{3423}\big)/27 \\
B^9_{3,5,1} &= \big(\triangle_{4611} + \triangle_{3621}\big)/3 + \big(\triangle_{4512} + \triangle_{3522}\big)/9 + \big(\triangle_{4413} + \triangle_{4323} + \triangle_{3423}\big)/27 \\
B^9_{3,1,5} &= \big(\triangle_{4161} + \triangle_{3261}\big)/3 + \big(\triangle_{4152} + \triangle_{3252}\big)/9 + \big(\triangle_{4143} + \triangle_{4233} + \triangle_{3243}\big)/27 \\
B^9_{1,4,4} &= \triangle_{1551}/3 + \big(\triangle_{1452} + \triangle_{1542} + \triangle_{2352} + \triangle_{2532}\big)/9 + 2\big(\triangle_{2343} + \triangle_{2433} + 2\triangle_{1443}\big)/27 \\
B^9_{4,1,4} &= \triangle_{5151}/3 + \big(\triangle_{4152} + \triangle_{5142} + \triangle_{3252} + \triangle_{5232}\big)/9 + 2\big(\triangle_{3243} + \triangle_{4233} + 2\triangle_{4143}\big)/27 \\
B^9_{4,4,1} &= \triangle_{5511}/3 + \big(\triangle_{4512} + \triangle_{5412} + \triangle_{3522} + \triangle_{5322}\big)/9 + 2\big(\triangle_{3423} + \triangle_{4323} + 2\triangle_{4413}\big)/27
\end{aligned} \tag{1.65}$$

$$
\begin{aligned}
B^9_{2,2,5} &= \big(\triangle_{2361} + \triangle_{3261}\big)/3 + \big(\triangle_{2352} + \triangle_{3252}\big)/9 \\
&\quad + \big(\triangle_{2343} + \triangle_{3243}\big)/27 + \big(\triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/81 \\
B^9_{2,5,2} &= \big(\triangle_{2631} + \triangle_{3621}\big)/3 + \big(\triangle_{2532} + \triangle_{3522}\big)/9 \\
&\quad + \big(\triangle_{2433} + \triangle_{3423}\big)/27 + \big(\triangle_{2334} + \triangle_{3324} + \triangle_{3234}\big)/81 \\
B^9_{5,2,2} &= \big(\triangle_{6231} + \triangle_{6321}\big)/3 + \big(\triangle_{5232} + \triangle_{5322}\big)/9 \\
&\quad + \big(\triangle_{4233} + \triangle_{4323}\big)/27 + \big(\triangle_{3234} + \triangle_{3324} + \triangle_{2334}\big)/81
\end{aligned}
\tag{1.66}
$$

$$
\begin{aligned}
B^9_{2,3,4} &= \big(\triangle_{1452} + 2\triangle_{2352} + \triangle_{3252} + \triangle_{2433}\big)/9 \\
&\quad + \big(2\triangle_{1443} + 4\triangle_{2343} + 2\triangle_{3243} + \triangle_{3423} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{2,4,3} &= \big(\triangle_{1542} + 2\triangle_{2532} + \triangle_{3522} + \triangle_{2343}\big)/9 \\
&\quad + 2\triangle_{1443} + 4\triangle_{2433} + 2\triangle_{3423} + \triangle_{3243} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{4,2,3} &= \big(\triangle_{5142} + 2\triangle_{5232} + \triangle_{5322} + \triangle_{3243}\big)/9 \\
&\quad + \big(2\triangle_{4143} + 4\triangle_{4233} + 2\triangle_{4323} + \triangle_{2343} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{4,3,2} &= \big(\triangle_{5412} + 2\triangle_{5322} + \triangle_{5232} + \triangle_{3423}\big)/9 \\
&\quad + \big(2\triangle_{4413} + 4\triangle_{4323} + 2\triangle_{4233} + \triangle_{2433} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{3,4,2} &= \big(\triangle_{4512} + 2\triangle_{3522} + \triangle_{2532} + \triangle_{4323}\big)/9 \\
&\quad + \big(2\triangle_{4413} + 4\triangle_{3423} + 2\triangle_{2433} + \triangle_{4233} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{3,2,4} &= \big(\triangle_{4152} + 2\triangle_{3252} + \triangle_{2352} + \triangle_{4233}\big)/9 \\
&\quad + \big(2\triangle_{4143} + 4\triangle_{3243} + 2\triangle_{2343} + \triangle_{4323} + \triangle_{2334} + \triangle_{3234} + \triangle_{3324}\big)/27 \\
B^9_{3,3,3} &= \big(\triangle_{2343} + \triangle_{2433} + \triangle_{3243} + \triangle_{3423} + \triangle_{4233} + \triangle_{4323}\big)/9 \\
&\quad + \big(\triangle_{1443} + \triangle_{4143} + \triangle_{4413} + 2\triangle_{2334} + 2\triangle_{3234} + 2\triangle_{3324}\big)/27
\end{aligned}
\tag{1.67}
$$

By (1.3)

$$
\big(u_1\beta_1 + u_2\beta_2 + u_3\beta_3\big)^9 = \sum_{(i,j,k)\in\mathcal{I}^3_{\text{Bernstein}}} u_1^i u_2^j u_3^k B^9_{ijk}(\boldsymbol{\beta}) + \sum_{(i,j,k)\in\mathcal{I}^3_{\text{removed}}} u_1^i u_2^j u_3^k B^9_{ijk}(\boldsymbol{\beta}).
$$

For $(i,j,k)\in\mathcal{I}^3_{\text{Bernstein}}$ we have $B^9_{i,j,k}(\boldsymbol{\beta}) = S^9_{i+1,j+1,k+1,0}(\boldsymbol{\beta})$ and hence $\rho_{i+1,j+1,k+1,0}(\boldsymbol{u}) = u_1^i u_2^j u_3^k$. In the second sum we insert the expressions in (1.65)–(1.67) for B^9_{ijk}, and collect terms for each $\triangle_{ijk\ell}$ to obtain (1.63). We show this for 7 typical cases. Let $\bar{u}_{r,s} := (u_r + u_s)/2$ for $r,s = 1,2,3$. Then

$$
\begin{aligned}
&\sum_{(i,j,k)\in\mathcal{I}^3_{\text{removed}}} u_1^i u_2^j u_3^k B^9_{ijk}(\boldsymbol{\beta}) = u_1^1u_2^4u_3^4\triangle_{1551}(\boldsymbol{\beta})/3 + \big(u_1^3u_2^1u_3^5 + u_1^2u_2^2u_3^5\big)\triangle_{3261}(\boldsymbol{\beta})/3 \\
&+ \big(u_1^5u_2^3u_3^1 + u_1^4u_2^4u_3^1 + u_1^4u_2^3u_3^2\big)\triangle_{5412}(\boldsymbol{\beta})/9 \\
&+ \big(u_1^3u_2^5u_3^1 + u_1^4u_2^4u_3^1 + u_1^2u_2^5u_3^2 + u_1^2u_2^4u_3^3 + 2u_1^3u_2^4u_3^2\big)\triangle_{3552}(\boldsymbol{\beta})/9 \\
&+ \big(u_1^5u_2^3u_3^1 + u_1^3u_2^5u_3^1 + 2u_1^4u_2^4u_3^1 + 2u_1^4u_2^3u_3^2 + u_1^3u_2^4u_3^2 + u_1^3u_2^3u_3^3\big)\triangle_{4413}(\boldsymbol{\beta})/27 \\
&+ \big(u_1^5u_2^1u_3^3 + u_1^3u_2^1u_3^5 + 2u_1^4u_2^1u_3^4 + u_1^5u_2^2u_3^2 + 4u_1^4u_2^2u_3^3 \\
&+ 2u_1^4u_2^3u_3^2 + u_1^3u_2^4u_3^2 + 3u_1^3u_2^2u_3^4 + 3u_1^3u_2^3u_3^3\big)\triangle_{4233}(\boldsymbol{\beta})/27 \\
&+ \big(u_1^2u_2^2u_3^5 + u_1^2u_2^5u_3^2 + u_1^5u_2^2u_3^2 + 3u_1^2u_2^3u_3^4 + 3u_1^2u_2^4u_3^3 + 3u_1^4u_2^2u_3^3 \\
&+ 3u_1^4u_2^3u_3^2 + 3u_1^3u_2^4u_3^2 + 3u_1^3u_2^2u_3^4 + 6u_1^3u_2^3u_3^3\big)\triangle_{3324}(\boldsymbol{\beta})/81 \;+\cdots \\
&= u_1^1u_2^4u_3^4\triangle_{1551}(\boldsymbol{\beta})/3 + +2u_1^2u_2^1u_3^5\bar{u}_{1,2}\triangle_{3261}(\boldsymbol{\beta})/3 + u_1^4u_2^3u_3^1\bar{u}_{123}\triangle_{5412}(\boldsymbol{\beta})/3 + 2u_1^2u_2^4u_3^1\bar{u}_{1,3}\bar{u}_{123}\triangle_{3522}(\boldsymbol{\beta})/3 \\
&+ u_1^3u_2^3u_3^1\bar{u}_{123}^2\triangle_{4413}(\boldsymbol{\beta})/3 + 2u_1^3u_2^1u_3^2\bar{u}_{2,3}\bar{u}_{123}^2\triangle_{4233}(\boldsymbol{\beta})/3 + u_1^2u_2^2u_3^2\bar{u}_{123}^3\triangle_{3324}(\boldsymbol{\beta})/3 \;+\cdots \\
&= u_1^1u_2^4u_3^4S^9_{1551}(\boldsymbol{\beta}) + u_1^2u_2^1u_3^5\bar{u}_{1,2}S^9_{3261}(\boldsymbol{\beta}) + u_1^4u_2^3u_3^1\bar{u}_{123}S^9_{5412}(\boldsymbol{\beta}) + u_1^2u_2^4u_3^1\bar{u}_{1,3}\bar{u}_{123}S^9_{3522}(\boldsymbol{\beta}) \\
&+ u_1^3u_2^3u_3^1\bar{u}_{123}^2S^9_{4413}(\boldsymbol{\beta}) + u_1^3u_2^1u_3^2\bar{u}_{2,3}\bar{u}_{123}^2S^9_{4233}(\boldsymbol{\beta}) + u_1^2u_2^2u_3^2\bar{u}_{123}^3S^9_{3324}(\boldsymbol{\beta}) \;+\cdots
\end{aligned}
$$

and (1.63) follows.

■

The barycentric form of the domain points are computed as explained in Corollary 1. For a plot see Figure 1.6.

For the stability, the computation is similar to the proof of Proposition 5, except that since $S^9_{3324}, S^9_{3234}, S^9_{2334}$ have the same domain points $(1/3, 1/3, 1/3)$. We replace this triple point by the three points $(3,3,1)/7$, $(3,1,3)/7$, $(1,3,3)/7$ and find

$$\kappa_{9,\infty}(\mathcal{T}) := \max_{\boldsymbol{c}\neq\boldsymbol{0}} \frac{\|\boldsymbol{b}^T\boldsymbol{c}\|_{L_\infty(\mathcal{T})}}{\|\boldsymbol{c}\|_\infty} \max_{\boldsymbol{c}\neq\boldsymbol{0}} \frac{\|\boldsymbol{c}\|_\infty}{\|\boldsymbol{b}^T\boldsymbol{c}\|_{L_\infty(\mathcal{T})}} \simeq 159\,844.34\ldots$$

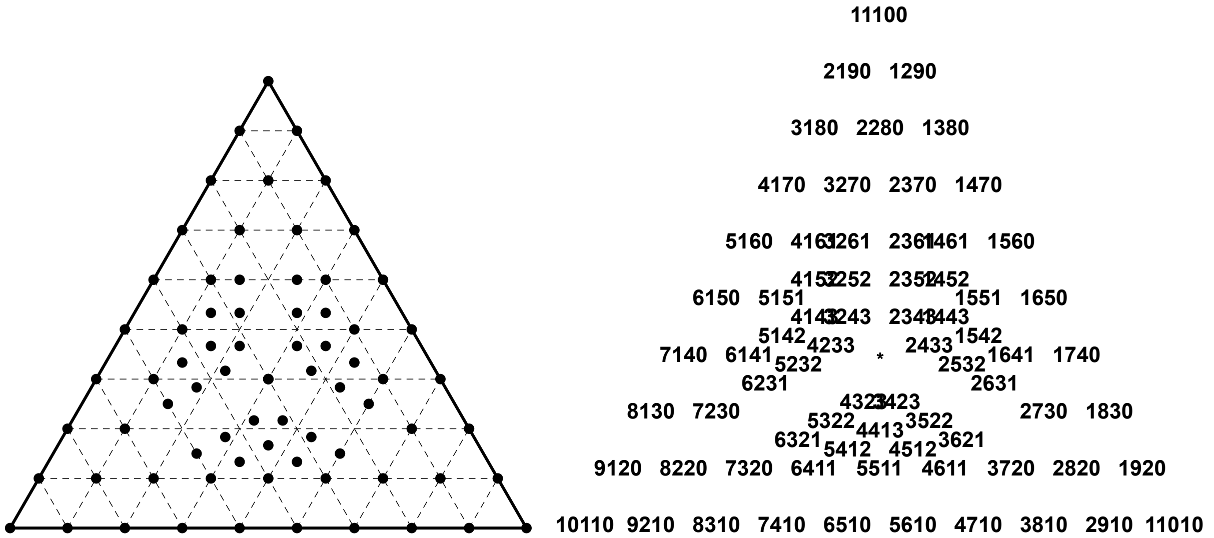


Fig. 1.6 Simplex splines domain points on the right, with their positions on the left. $S^9_{3324}, S^9_{3234}, S^9_{2334}$ have the same domain point $(1/3, 1/3, 1/3)$ as indicated by a *

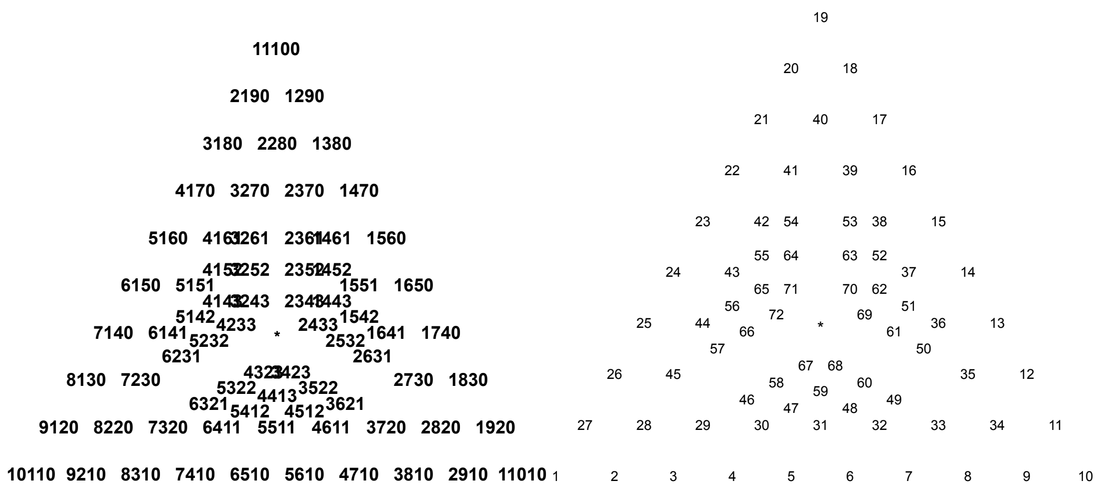


Fig. 1.7 Simplex splines domain points on the left, with their sorting on the left. $S^9_{3324}, S^9_{3234}, S^9_{2334}$ indicated by a * have the numbers 73,74,75

C^3–continuity through an edge:

Let $s = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^3} c_{ijk\ell} S^9_{ijk\ell}$ (respectively $\tilde{s} = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^3} \tilde{c}_{ijk\ell} \tilde{S}^9_{ijk\ell}$) be defined on the triangle $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\rangle$ (resp. $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3\rangle$). We suppose that $\tilde{\boldsymbol{p}}_3 = \lambda_1\boldsymbol{p}_1 + \lambda_2\boldsymbol{p}_2 + \lambda_3\boldsymbol{p}_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

With an extension of the notations of Proposition 6, we define

$$\begin{aligned}
\boldsymbol{c}_0 &= [c_{10-i,i+1,1,0}]_{i=0,\dots,9}^T,\\
\boldsymbol{c}_1 &= [c_{9120}, c_{8220}, c_{7320}, c_{6411}, c_{5511}, c_{4611}, c_{3720}, c_{2820}, c_{1920}]^T,\\
\boldsymbol{c}_2 &= [c_{8130}, c_{7230}, c_{6321}, c_{7412}, c_{4512}, c_{3621}, c_{2730}, c_{1830}]^T,\\
\boldsymbol{c}_3 &= [c_{7140}, c_{6231}, c_{5322}, c_{4413}, c_{43522}, c_{2631}, c_{1740}]^T.
\end{aligned}$$

For $r \leq 3$, we connect with smoothness C^r two adjacent triangles in the following proposition.

Proposition 7 *Let* $s = \sum_{(i,j,k,\ell)\in\bar{I}^2} c_{ijk\ell} S^9_{ijk\ell}$ *(respectively* $\tilde{s} = \sum_{(i,j,k,\ell)\in\bar{I}^2} \tilde{c}_{ijk\ell} \tilde{S}^9_{ijk\ell}$ *) be defined on the triangle* $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ *(resp.* $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3 \rangle$*). We suppose that* $\tilde{\boldsymbol{p}}_3 = \lambda_1 \boldsymbol{p}_1 + \lambda_2 \boldsymbol{p}_2 + \lambda_3 \boldsymbol{p}_3$ *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$.
The function $s_+ = \begin{cases} s & \text{on } \mathcal{T} \\ \tilde{s} & \text{on } \tilde{\mathcal{T}} \end{cases}$ *is* C^r *with* $r \leq 3$ *if and only if*

$$[\tilde{\boldsymbol{c}}_m]_{m=0,\dots,r} = [\boldsymbol{C}_{mn}]_{m,n=0,\dots,r} [\boldsymbol{c}_n]_{n=0,\dots,n}, \quad \boldsymbol{C}_{mn} = \in \mathbb{R}^{(10-m)\times(10-n)} \tag{1.68}$$

where the nonzero submatrices or components are written below in (1.69) (1.70)*,* (1.71) *and* (1.72)*.*

The proof of the proposition is a reproduction of the one of Proposition 6, firstly by connecting two polynomials written in the two corresponding Bernstein basis, then computing the Bernstein polynomials in the Simplex-Splines basis.

$$\boldsymbol{C}_{00} = \boldsymbol{I} \in \mathbb{R}^{10\times10}, \tag{1.69}$$

$$\boldsymbol{C}_{10} = \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_1 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_1 & \lambda_2 \end{bmatrix} \in \mathbb{R}^{9\times10}, \quad \boldsymbol{C}_{11} = \lambda_3 \boldsymbol{I} \in \mathbb{R}^{9\times9}, \tag{1.70}$$

$$\begin{array}{lcccc}
\boldsymbol{C}_{20}(1,1:3) &=& \lambda_1^2 & 2\lambda_1\lambda_2 & {\lambda_2}^2 \\
\boldsymbol{C}_{20}(2,2:4) &=& \lambda_1^2 & 2\lambda_1\lambda_2 & {\lambda_2}^2 \\
\boldsymbol{C}_{20}(3,3:5) &=& \frac{\lambda_1^2}{2} & \lambda_1\lambda_2 + \frac{\lambda_1}{2} & \frac{\lambda_2^2}{2} + \frac{\lambda_2}{2} \\
\boldsymbol{C}_{20}(4,4:6) &=& \frac{\lambda_1^2}{3} + \frac{\lambda_1}{3} & \frac{2\lambda_1\lambda_2}{3} + \frac{\lambda_2}{3} + \frac{\lambda_1}{3} & \frac{\lambda_2^2}{3} + \frac{\lambda_2}{3} \\
\boldsymbol{C}_{20}(5,5:7) &=& \frac{\lambda_1^2}{3} + \frac{\lambda_1}{3} & \frac{2\lambda_1\lambda_2}{3} + \frac{\lambda_2}{3} + \frac{\lambda_1}{3} & \frac{\lambda_2^2}{3} + \frac{\lambda_2}{3} \\
\boldsymbol{C}_{20}(6,6:8) &=& \frac{\lambda_1^2}{2} + \frac{\lambda_1}{2} & \lambda_1\lambda_2 + \frac{\lambda_2}{2} & \frac{\lambda_2^2}{2} \\
\boldsymbol{C}_{20}(7,7:9) &=& \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2 \\
\boldsymbol{C}_{20}(8,8:10) &=& \lambda_1^2 & 2\lambda_1\lambda_2 & \lambda_2^2
\end{array}$$

$$\left.\begin{array}{lccc}
\boldsymbol{C}_{21}(1,1:2) &=& 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 \\
\boldsymbol{C}_{21}(2,2:3) &=& 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 \\
\boldsymbol{C}_{21}(3,3:4) &=& \lambda_1\lambda_3 & \frac{2\lambda_2\lambda_3-\lambda_3^2}{2} + \frac{\lambda_3}{2} \\
\boldsymbol{C}_{21}(4,4:5) &=& \frac{2\lambda_1\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{3} & \frac{2\lambda_2\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{3} \\
\boldsymbol{C}_{21}(5,5:6) &=& \frac{2\lambda_1\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{3} & \frac{2\lambda_2\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{3} \\
\boldsymbol{C}_{21}(6,6:7) &=& \frac{2\lambda_1\lambda_3-\lambda_3^2}{2} + \frac{\lambda_3}{2} & \lambda_2\lambda_3 \\
\boldsymbol{C}_{21}(7,7:8) &=& 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 \\
\boldsymbol{C}_{21}(8,8:9) &=& 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3
\end{array}\right., \quad \boldsymbol{C}_{22} = \lambda_3^2 \boldsymbol{I} \in \mathbb{R}^{8\times8}, \tag{1.71}$$

$$\begin{array}{lccccc}
\boldsymbol{C}_{30}(1,1:4) &=& \lambda_1^3 & 3\lambda_1^2\lambda_2 & 3\lambda_1\lambda_2^2 & \lambda_2^3 \\
\boldsymbol{C}_{30}(2,2:5) &=& \frac{\lambda_1^3}{2} & \frac{3\lambda_1^2\lambda_2}{2} + \frac{\lambda_1^2}{2} & \frac{3\lambda_1\lambda_2^2}{2} + \lambda_1\lambda_2 & \frac{\lambda_2^3}{2} + \frac{\lambda_2^2}{2} \\
\boldsymbol{C}_{30}(3,3:6) &=& \frac{\lambda_1^3}{6} + \frac{\lambda_1^2}{6} & \frac{\lambda_1^2\lambda_2}{2} + \frac{\lambda_1\lambda_2}{3} + \frac{\lambda_1^2}{3} + \frac{\lambda_1}{6} & \frac{\lambda_1\lambda_2^2}{2} + \frac{\lambda_2^2}{6} + \frac{2\lambda_1\lambda_2}{3} + \frac{\lambda_2}{6} + \frac{\lambda_1}{6} & \frac{\lambda_2^3}{6} + \frac{\lambda_2^2}{3} + \frac{\lambda_2}{6} \\
\boldsymbol{C}_{30}(4,4:7) &=& \frac{\lambda_1^3}{9} + \frac{2\lambda_1^2}{9} + \frac{\lambda_1}{9} & \frac{\lambda_1^2\lambda_2}{3} + \frac{4\lambda_1\lambda_2}{9} + \frac{\lambda_2}{9} + \frac{2\lambda_1^2}{9} + \frac{2\lambda_1}{9} & \frac{\lambda_1\lambda_2^2}{3} + \frac{2\lambda_2^2}{9} + \frac{4\lambda_1\lambda_2}{9} + \frac{2\lambda_2}{9} + \frac{\lambda_1}{9} & \frac{\lambda_2^3}{9} + \frac{2\lambda_2^2}{9} + \frac{\lambda_2}{9} \\
\boldsymbol{C}_{30}(5,5:8) &=& \frac{\lambda_1^3}{6} + \frac{\lambda_1^2}{3} + \frac{\lambda_1}{6} & \frac{\lambda_1^2\lambda_2}{2} + \frac{2\lambda_1\lambda_2}{3} + \frac{\lambda_2}{6} + \frac{\lambda_1^2}{6} + \frac{\lambda_1}{6} & \frac{\lambda_1\lambda_2^2}{2} + \frac{\lambda_2^2}{3} + \frac{\lambda_1\lambda_2}{3} + \frac{\lambda_2}{6} & \frac{\lambda_2^3}{6} + \frac{\lambda_2^2}{6} \\
\boldsymbol{C}_{30}(6,6:9) &=& \frac{\lambda_1^3}{2} + \frac{\lambda_1^2}{2} & \frac{3\lambda_1^2\lambda_2}{2} + \lambda_1\lambda_2 & \frac{3\lambda_1\lambda_2^2}{2} + \frac{\lambda_2^2}{2} & \frac{\lambda_2^3}{2} \\
\boldsymbol{C}_{30}(7,7:10) &=& \lambda_1^3 & 3\lambda_1^2\lambda_2 & 3\lambda_1\lambda_2^2 & \lambda_2^3
\end{array} \tag{1.72}$$

$$
\begin{array}{lll}
\boldsymbol{C}_{31}(1,1:2) = & 3\lambda_1^2\lambda_3 & 6\lambda_1\lambda_2\lambda_3 \\
\boldsymbol{C}_{31}(1,3) \quad = & 3\lambda_2^2\lambda_3 & \\
\boldsymbol{C}_{31}(2,2:3) = & \frac{3\lambda_1^2\lambda_3}{2} & 3\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3 \\
\boldsymbol{C}_{31}(2,4) \quad = & \frac{\lambda_3^3-3\lambda_2\lambda_3^2+3\lambda_2^2\lambda_3}{2} + \frac{2\lambda_2\lambda_3-\lambda_3^2}{2} & \\
\boldsymbol{C}_{31}(3,3:4) = & \frac{\lambda_1^2\lambda_3}{2} + \frac{\lambda_1\lambda_3}{3} & \frac{2\lambda_3^3-3\lambda_2\lambda_3^2-3\lambda_1\lambda_3^2+6\lambda_1\lambda_2\lambda_3}{6} + \frac{2\lambda_2\lambda_3-\lambda_3^2}{6} + \frac{2\lambda_1\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{6} \\
\boldsymbol{C}_{31}(3,5) \quad = & \frac{\lambda_3^3-3\lambda_2\lambda_3^2+3\lambda_2^2\lambda_3}{6} + \frac{2\lambda_2\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{6} & \\
\boldsymbol{C}_{31}(4,4:5) = & \frac{\lambda_3^3-3\lambda_1\lambda_3^2+3\lambda_1^2\lambda_3}{9} + \frac{2\left(2\lambda_1\lambda_3-\lambda_3^2\right)}{9} + \frac{\lambda_3}{9} & \frac{2\lambda_3^3-3\lambda_2\lambda_3^2-3\lambda_1\lambda_3^2+6\lambda_1\lambda_2\lambda_3}{9} + \frac{2\left(2\lambda_2\lambda_3-\lambda_3^2\right)}{9} + \frac{2\left(2\lambda_1\lambda_3-\lambda_3^2\right)}{9} + \frac{2\lambda_3}{9} \\
\boldsymbol{C}_{31}(4,6) \quad = & \frac{\lambda_3^3-3\lambda_2\lambda_3^2+3\lambda_2^2\lambda_3}{9} + \frac{2\left(2\lambda_2\lambda_3-\lambda_3^2\right)}{9} + \frac{\lambda_3}{9} & \\
\boldsymbol{C}_{31}(5,5:6) = & \frac{\lambda_3^3-3\lambda_1\lambda_3^2+3\lambda_1^2\lambda_3}{6} + \frac{2\lambda_1\lambda_3-\lambda_3^2}{3} + \frac{\lambda_3}{6} & \frac{2\lambda_3^3-3\lambda_2\lambda_3^2-3\lambda_1\lambda_3^2+6\lambda_1\lambda_2\lambda_3}{6} + \frac{2\lambda_2\lambda_3-\lambda_3^2}{3} + \frac{2\lambda_1\lambda_3-\lambda_3^2}{6} + \frac{\lambda_3}{6} \\
\boldsymbol{C}_{31}(5,7) \quad = & \frac{\lambda_2^2\lambda_3}{2} + \frac{\lambda_2\lambda_3}{3} & \\
\boldsymbol{C}_{31}(6,6:7) = & \frac{\lambda_3^3-3\lambda_1\lambda_3^2+3\lambda_1^2\lambda_3}{2} + \frac{2\lambda_1\lambda_3-\lambda_3^2}{2} & \\
\boldsymbol{C}_{31}(6,8) \quad = & 3\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3 & \frac{3\lambda_2^2\lambda_3}{2} \\
\boldsymbol{C}_{31}(7,7:8) = & 3\lambda_1^2\lambda_3 & 6\lambda_1\lambda_2\lambda_3 \\
\boldsymbol{C}_{31}(7,9) \quad = & 3\lambda_2^2\lambda_3 &
\end{array}
$$

$$
\begin{array}{lll}
\boldsymbol{C}_{32}(1,1:2) = & 3\lambda_1\lambda_3^2 & 3\lambda_2\lambda_3^2 \\
\boldsymbol{C}_{32}(2,2:3) = & \frac{3\lambda_1\lambda_3^2}{2} & \frac{6\lambda_2\lambda_3^2-2\lambda_3^3}{2} + \lambda_3^2 \\
\boldsymbol{C}_{32}(3,3:4) = & \frac{6\lambda_1\lambda_3^2-2\lambda_3^3}{6} + \frac{\lambda_3^2}{3} & \frac{9\lambda_2\lambda_3^2-6\lambda_3^3}{6} + \lambda_3^2 \\
\boldsymbol{C}_{32}(4,4:5) = & \frac{9\lambda_1\lambda_3^2-6\lambda_3^3}{9} + \frac{2\lambda_3^2}{3} & \frac{9\lambda_2\lambda_3^2-6\lambda_3^3}{9} + \frac{2\lambda_3^2}{3} \\
\boldsymbol{C}_{32}(5,5:6) = & \frac{9\lambda_1\lambda_3^2-6\lambda_3^3}{6} + \lambda_3^2 & \frac{6\lambda_2\lambda_3^2-2\lambda_3^3}{6} + \frac{\lambda_3^2}{3} \\
\boldsymbol{C}_{32}(6,6:7) = & \frac{6\lambda_1\lambda_3^2-2\lambda_3^3}{2} + \lambda_3^2 & \frac{3\lambda_2\lambda_3^2}{2} \\
\boldsymbol{C}_{32}(7,7:8) = & 3\lambda_1\lambda_3^2 & 3\lambda_2\lambda_3^2
\end{array}
\qquad \boldsymbol{C}_{33} = \lambda_3^3\boldsymbol{I} \in \mathbb{R}^{7\times 7}
$$

1.4.3 Conclusion

For any $r \geq 1$, we have built a B-spline like basis made out of simplex splines for the space $\mathbb{S}_{3r}^r(\triangle)$ of splines on the Clough-Tocher split $\triangle$ on a single triangle. For even values of r, we removed one of the elements in order to obtain the partition of unity and a Marsden-like identity proved for $r \leq 3$ and shown symbolically for $r \leq 6$. Looking in more detail at the cases $r = 2, 3$, corresponding to degrees $d = 6, 9$, we gave explicit formulas for connecting two neighbooring triangles in a C^r fashion across an edge using Bernstein-Bézier techniques, and gave an upper bound for the L_∞ condition number of the basis.

For $r = 4, 5, 6$ the domain points can be computed as for $r \leq 3$ using the Marsden like-identity shown below, which gives interpolation points to study the stability . The coefficients to obtain the C^r**–connection** between two triangles $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ and $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3 \rangle$ can be found by a computation in the Bernstein basis of the polynomials in $\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_{\boldsymbol{T}} \rangle$ and $\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_{\boldsymbol{T}} \rangle$.

We end by restating the Marsden-like identity in a general form. It is proved for $r = 1, 2, 3$ and symbolically for $r = 4, 5, 6$. It is a conjecture for $r > 6$.

Theorem 4 (The barycentric Marsden-like identity for degree $3r$**,** $r \leq 6$**.)**
For $r \in \mathbb{N}$, $d = 3r$, $u_i, \beta_i \in \mathbb{R}$, *with* $\beta_i \geq 0$, $i = 1, 2, 3$, *and* $\beta_1 + \beta_2 + \beta_3 = 1$ *we have*

$$(u_1\beta_1 + u_2\beta_2 + u_3\beta_3)^d = \sum_{(i,j,k,\ell)\in\bar{\mathcal{I}}^r} \rho_{ijk\ell}(u_1, u_2, u_3) S_{ijk\ell}^d(\beta_1, \beta_2, \beta_3),$$

where the index set $\bar{\mathcal{I}}^r$ *is given in Definition 3 and*

$$S_{ijk\ell}^{3r} := c_{ijk\ell} \triangle_{ijk\ell}, \text{ where } c_{ijk\ell} := \begin{cases} 1, & \text{if } \ell = 0, \\ 2/3 & \text{if } \max(\epsilon_1, \epsilon_2, \epsilon_3) = 1/2, \\ 1/3 & \text{otherwise}, \end{cases} \tag{1.73}$$
$$\rho_{ijk\ell}(u_1, u_2, u_3) := u_1^{i-1} u_2^{j-1} u_3^{k-1} \bar{u}_{123}^{\lambda-1} (\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \delta_{0\ell})^{1-\delta_{0\ell}}.$$

Here, $\lambda := \max(\ell, 1)$, $\quad \bar{u}_{123} := (u_1 + u_2 + u_3)/3$, $\quad \nu := \max(i, j, k)$, *and*

$$\epsilon_1 := \begin{cases} 1/\gamma_1, & i < \nu \;\&\; j \neq 1 \;\&\; k \neq 1, \\ 0, & otherwise \end{cases}, \quad \gamma_1 := \begin{cases} 2, & i \neq 1 \;\&\; j \neq k, \\ 1, & otherwise\,, \end{cases}$$
$$\epsilon_2 := \begin{cases} 1/\gamma_2, & j < \nu \;\&\; i \neq 1 \;\&\; k \neq 1, \\ 0, & otherwise \end{cases}, \quad \gamma_2 := \begin{cases} 2, & j \neq 1 \;\&\; i \neq k, \\ 1, & otherwise\,, \end{cases} \tag{1.74}$$
$$\epsilon_3 := \begin{cases} 1/\gamma_3, & k < \nu \;\&\; i \neq 1 \;\&\; j \neq 1, \\ 0, & otherwise \end{cases}, \quad \gamma_3 := \begin{cases} 2, & k \neq 1 \;\&\; i \neq j, \\ 1, & otherwise\,. \end{cases}$$

Proof For $r = 1, 2, 3$ this is an alternative way of formulating Theorem 5 in [19] for $r = 1$, and Theorems 2,3. To see this consider first $r = 1$. In [19] it was shown that

$$\begin{aligned}(\boldsymbol{\beta}^T \boldsymbol{u})^3 &= u_1^3 S_1(\boldsymbol{\beta}) + u_1^2 u_2 S_2(\boldsymbol{\beta}) + u_1 u_2^2 S_3(\boldsymbol{\beta}) + u_2^3 S_4(\boldsymbol{\beta}) + u_2^2 u_3 S_5(\boldsymbol{\beta}) \\ &+ u_2 u_3^2 S_6(\boldsymbol{\beta}) + u_3^3 S_7(\boldsymbol{\beta}) + u_1 u_3^2 S_8(\boldsymbol{\beta}) + u_1^2 u_3 S_9(\boldsymbol{\beta}) \\ &+ u_1 u_2 u_3 \big(S_{10}(\boldsymbol{\beta}) + S_{11}(\boldsymbol{\beta}) + S_{12}(\boldsymbol{\beta})\big),\end{aligned} \tag{1.75}$$

where $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3)$ and $\boldsymbol{u} := (u_1, u_2, u_3)$. For the first 9 (Bernstein) dual functions we find $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, $\delta_{0\ell} = 1$, and (1.73) holds. For the dual function $\rho[1221]$ corresponding to $S_{10} = \triangle_{1221}/3$ we find $(\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \delta_{0\ell})^{1-\delta_{0\ell}} = u_1$ and (1.73) gives $\rho_{1221}(\boldsymbol{u}) S_{1221}^3(\boldsymbol{\beta}) = u_1 u_2 u_3 S_{10}(\boldsymbol{\beta})$ as stated in (1.75). The results for $\rho[2121]$ and $\rho[2211]$ are similar. For $r = 2, 3$ it follows as for $r = 1$ that (1.73) holds for the Bernstein polynomials in $\mathcal{I}^2_{\text{Bernstein}}$. For $r = 2$ consider for example $\rho_{2322}(\boldsymbol{u}) = u_1 u_2^2 u_3 \bar{u}_{1,3} \bar{u}_{123}$ and $S_{2322} = 2\triangle_{2332}/3$ in Theorem 2. This is the same as the expressions in (1.73) since $\epsilon_1 = \epsilon_3 = 1/2$ and $\epsilon_3 = 0$. As an example for $r = 3$, $\rho_{3324}(\boldsymbol{u}) = u_1^2 u_2^2 u_3^2 \bar{u}_{123}^3$ and $S_{3324} = \triangle_{3324}/3$ in Theorem 3 is the same as the expressions in (1.73) since $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_3 = 1$. All other cases are verified similarly.
■

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1.5 Appendix

We provide the reader with free extra entertainment by giving alternative proofs, not facts, for some properties of our spline basis.

1.5.1 Generating system for the spline space

To show that Σ_{3r}^r generates the space of all simplex splines in, $\mathbb{S}_{3r}^r(\triangle)$, we begin with a definition and a lemma.

Definition 5 We introduce, for $m \geq 0$ and $\mu \leq d + 1$ the spaces

$$\begin{aligned} M_m &= \operatorname{span}\left\{\triangle[i, j, k; \ell] \in \mathbb{S}_d^r(\triangle) : \ \ell \geq m, \max\{i, j, k\} = \mu - \ell\right\}, \\ W_m &= \operatorname{span}\left\{\triangle[i, j, k; \ell] \in \mathbb{S}_d^r(\triangle) : \ \ell \geq m, \min\{i, j, k\} = 1\right\}. \end{aligned}$$

Note that by definition $M_m \supseteq M_{m+1}$ and $W_m \supseteq W_{m+1}$ for any $m \in \mathbb{N}_0$.

Lemma 5 *Let* $\ell \in \mathbb{N}_0$ *and* $i, j, k \in \mathbb{N}$ *with* $i + j + k + \ell = d + 3$. *If* $\max\{i, j, k\} < \mu - \ell$ *and* $\min\{i, j, k\} > 1$ *then* $\triangle[i, j, k; \ell] \in M_{\ell+1} + W_{\ell+1}$.

Proof We prove the lemma by induction on the *defect* $\nu = \mu - \ell - \max\{i, j, k\}$ which starts at $\nu = 1$ since $\max\{i, j, k\} < \mu - \ell$.

If $\nu = 1$, suppose without loss of generality that $\max\{i,j,k\} = i$, otherwise one can permute the knots appropriately. The knot insertion formula (1.18) yields that

$$\triangle[i,j,k;\ell] = \frac{1}{3}\left(\triangle[i-1,j,k;\ell+1] + \triangle[i,j-1,k;\ell+1] + \triangle[i,j,k-1;\ell+1]\right), \tag{1.76}$$

and the two functions $\triangle[i,j-1,k;\ell+1]$ and $\triangle[i,j,k-1;\ell+1]$ satisfy

$$\max\left\{\begin{array}{l}\{i,j-1,k\}\\ \{i,j,k-1\}\end{array}\right\} = i = \mu - \ell - \nu = \mu - (\ell+1),$$

hence belong to Σ_{3r}^r, thus to $M_{\ell+1}$. The same holds true for $\triangle[i-1,j,k;\ell+1]$ if $j = i$ or $k = i$. In the remaining case, $i > \max\{j,k\}$, we decompose

$$\triangle[i-1,j,k;\ell+1] = \frac{1}{3}\left(\triangle[i-2,j,k;\ell+2] + \triangle[i-1,j-1,k;\ell+2] + \triangle[i-1,j,k-1;\ell+2]\right)$$

and note that the last two functions on the right hand side again belong to Σ_{3r}^r, thus to $M_{\ell+2} \subset M_{\ell+1}$, so that again one only has to look at the first term. This procedure is repeated $n := i - \max\{j,k\}$ times when $i - n = \max\{j,k\}$ and thus $\triangle[i-n,j,k;\ell+n] \in M_{\ell+n} \subset M_{\ell+1}$.

To advance the induction, suppose that the result has been verified for some defect $\nu \geq 1$ and again apply the decomposition (1.76) to a spline in Σ_{3r}^r of defect $\nu+1$. Suppose again without loss of generality that $i = \max\{i,j,k\}$. We begin by looking at the first element of the decomposition where three things can happen:

1. $i = 2$, then immediately $\triangle[i-1,j,k;\ell+1] \in W_{\ell+1}$,
2. the defect is ν if $i = j$ or $i = k$, and the hypothesis yields that $\triangle[i-1,j,k;\ell+1] \in M_{\ell+1} + W_{\ell+1}$,
3. still we have $\nu+1$ which happens if $i > \max\{j,k\}$. In this case, we repeat the above argument of $n := i - \max\{j,k\}$ iterated decompositions until, eventually, $\triangle[i-n,j,k;\ell+n] \in M_{\ell+n} + W_{\ell+n} \subset M_{\ell+1} + W_{\ell+1}$. □

For the two other elements of the decomposition, the defect of $\triangle[i,j-1,k;\ell+1]$ and $\triangle[i,j,k-1;\ell+1]$ is

$$\mu - (\ell+1) - \max\{i,j-1,j,k-1,k\} = \mu - (\ell+1) - i = (\mu - \ell - \max\{i,j,k\}) - 1 = \nu + 1 - 1 = \nu$$

and the induction hypothesis yields that

$$\left.\begin{array}{l}\triangle[i,j-1,k;\ell+1]\\ \triangle[i,j,k-1;\ell+1]\end{array}\right\} \in M_{\ell+2} + W_{\ell+2} \subset M_{\ell+1} + W_{\ell+1}.$$

This advances the induction hypothesis and completes the proof of the lemma.
■

Proposition 8 Σ_{3r}^r *generates the space of all simplex splines in* $\mathbb{S}_{3r}^r$.

Proof First recall that, by assumption, $\mu = 3r + 1 - r = 2r + 1$ and let $\triangle[i,j,k;\ell]$ be one of the simplex spline generating $\mathbb{S}_{3r}^r(\triangle)$, which implies that

1. $\min\{i,j,k,\ell\} \geq 0$ and $i + j + k + \ell = 3r + 3$,
2. $\ell > 0$ implies $\max\{i,j,k\} \leq \mu - \ell = 2r + 1 - \ell$ for C^r–smoothness,
3. if $\min\{i,j,k\} = 0$, then $\ell > 0$ since the Bernstein polynomials with zero multiplicity correspond to distributions defined only on the boundary of the simplex.

We distinguish three cases:

1. $\max\{i,j,k\} > \mu - \ell$, then $\ell = 0$ and $\triangle[i,j,k;\ell]$ is already an element of Type (2),
2. $\max\{i,j,k\} = \mu - \ell$ then

   a. if $i := \min\{i,j,k\} = 0$ and $j := \max\{i,j,k\} = \mu - \ell = 2r + 1 - \ell$. Since $i + j + k + \ell = 3r + 3$, we deduce that $k = 2 + r \geq 2$. By knot insertion,

   $$\triangle[0,j,k;\ell] = \frac{1}{3}\left(\triangle[1,j-1,k;\ell] + \triangle[1,j,k-1;\ell] + \triangle[1,j,k;\ell-1]\right).$$

   The first element in this expression, $\triangle[1,j-1,k;\ell]$ is of Type (3) if $k < j$ and of Type (1) if $k = j$, the second one, $\triangle[1,j,k-1;\ell]$ is always of Type (1) and the third one, $\triangle[1,j,k;\ell-1]$, of Type (3). Consequently, $\triangle[0,j,k;\ell] \in \mathrm{span}(\Sigma_{3r}^r)$.

b. if $\min\{i,j,k\} \geq 1$, then $\triangle[i,j,k;\ell]$ is already an element of Type (1), hence in Σ^r_{3r}.

3. $\max\{i,j,k\} < \mu - \ell$

a. if $i := \min\{i,j,k\} = 0$, assume that $j := \max\{i,j,k\}$. Since $i + j + k + \ell = 3r + 3$, we deduce that $j \geq k > 2 + r \geq 2$ and knot insertion yields

$$\triangle[0,j,k;\ell] = \frac{1}{3}\left(\triangle[1,j-1,k;\ell] + \triangle[1,j,k-1;\ell] + \triangle[1,j,k;\ell-1]\right)$$

where the three elements on the right hand side are all of Type (3), hence $\triangle[0,j,k;\ell] \in \mathrm{span}(\Sigma^r_{3r})$.

b. if $\min\{i,j,k\} = 1$, then $\triangle[i,j,k;\ell]$ is immediately an element of Type (3).

c. if $\min\{i,j,k\} > 1$ we refer to Lemma 5 and find that $\triangle[i,j,k;\ell] \in M_{\ell+1} + W_{\ell+1} \subset M_0 + W_0$.

To finish the proof and to complete the last of these cases, we have to show that $M_0 + W_0 \subseteq \mathrm{span}(\Sigma^r_{3r})$. In fact, $M_0 \subset \mathrm{span}(\Sigma^r_{3r})$ was exactly proved in case 2. To complete the proof, let $\triangle[i,j,k;\ell]$ be a generator of W_0. If $\max\{i,j,k\} > \mu - \ell$ then once more $\ell = 0$ and $\triangle[i,j,k;\ell] = \triangle[i,j,k;0]$ is an element of Type (2). If, on the other hand, $\max\{i,j,k\} = \mu - \ell$, then $\triangle[i,j,k;\ell]$ is an element of Type (1), and if $\max\{i,j,k\} < \mu - \ell$, then $\triangle[i,j,k;\ell]$ is an element of Type (3). Consequently, $\triangle[i,j,k;\ell] \in \Sigma^r_{3r}$ and thus $W_0 \subset \mathrm{span}(\Sigma^r_{3r})$ which completes the proof that

$$M_0 + W_0 \subseteq \mathrm{span}(\Sigma^r_{3r}) \tag{1.77}$$

and the proof of the theorem.

■

1.5.2 A special case of linear independence

Here we give a direct proof of the fact stated in Proposition 4 that the functions in $\bar{\Sigma}^2_6$ are linearly independent on $\triangle$.

Proof Suppose for some real numbers $\{c_j\}$ that $\sum_{j=1}^{36} c_j S^6_j(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \triangle$. We first show that $c_j = 0$ for $j = 1,\ldots,18$. These corresponds to Bernstein polynomials with domainpoints on the boundary of $\triangle$. Consider the edge $\langle \boldsymbol{p}_1, \boldsymbol{p}_2\rangle$ corresponding to $\beta_3 = 0$. Looking at Figures 1.3, 1.4 we see that $i + j \leq 7$ for $S^6_8,\ldots,S^6_{36}$. By the local smoothness property only $S^6_1,\ldots,S^6_7$ can be nonzero on this edge. Moreover, on the same edge, these functions reduce to linearly independent univariate Bernstein polynomials B^6_{ij} for $i + j = 6$. It follows that $c_1 = \cdots = c_7 = 0$. With similar arguments on the edges $\langle \boldsymbol{p}_2, \boldsymbol{p}_3\rangle$ and $\langle \boldsymbol{p}_3, \boldsymbol{p}_1\rangle$ we conclude that $c_j = 0$ for $j \leq 18$.

The remaining simplex splines S^6_j $j = 19,\ldots,36$ are located on 3 rings. On ring k we find S^6_j for $j = 19,\ldots,27$ for $k = 1$, $j = 28,\ldots,33$ for $k = 2$, and $j = 34, 35, 36$ for $k = 3$, see Figure 1.4. On the horizontal part of these rings we take partial derivatives of order k with respect to β_3 and evaluate at $\beta_3 = 0$. On the parts parallel to $\langle \boldsymbol{p}_2, \boldsymbol{p}_3\rangle$ we take partial derivatives of order k with respect to β_1 and evaluate at $\beta_1 = 0$. Similarly we use β_2 on the last parts. The details are as follows

The horizontal part of the first inner ring contains the functions $S^6_{19}, S^6_{22}, S^6_{23}, S^6_{20}$ corresponding to $S^6_{ijk\ell}$ with $i + j = 7$. By (1.47) $S^6_j|_{\mathcal{T}_3} = O(\beta_3^2)$ for $j = 19,\ldots,36$, $j \neq 19, 20, 22, 23$ and

$$S^6_{19}|_{\mathcal{T}_3} = 30\beta_1^4\beta_2\beta_3, \quad S^6_{22}|_{\mathcal{T}_3} = 60\beta_1^3\beta_2^2\beta_3 + O(\beta_3^2) \quad S^6_{23}|_{\mathcal{T}_3} = 60\beta_1^2\beta_2^3\beta_3 + O(\beta_3^2), \quad S^6_{20}|_{\mathcal{T}_3} = 30\beta_1\beta_2^4\beta_3.$$

With $\boldsymbol{x} = (\beta_1, \beta_2, 0)$ we then find

$$\begin{aligned} 0 &= \frac{\partial}{\partial\beta_3}\left(\sum_{j=19}^{36} c_j S^6_j\right)(\boldsymbol{x}) = \frac{\partial}{\partial\beta_3}\left(c_{19}S^6_{19} + c_{23}S^6_{22} + c_{23}S^6_{23} + c_{20}S^6_{20}\right)(\boldsymbol{x}) \\ &= 30c_{19}\beta_1^4\beta_2 + 60c_{22}\beta_1^3\beta_2^2 + 60c_{23}\beta_1^2\beta_2^3 + 30c_{20}\beta_1\beta_2^4, \end{aligned}$$

a linear combination of linearly independent univariate Bernstein polynomials of degree 5. It follows that $c_{19} = c_{22} = c_{23} = c_{20} = 0$. With a similar argument with $S^6_{24}, S^6_{25}, S^6_{21}$ on $\mathcal{T}_1$ and S^6_{26}, S^6_{27} on $\mathcal{T}_2$ we conclude that $c_j = 0$ for $j \leq 27$.

Moving to the next ring we consider $S^6_{28}, S^6_{29}, S^6_{30}$ on the horizontal part and obtain from (1.46) and (1.47) with $\boldsymbol{x} = (\beta_1, \beta_2, 0)$

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial \beta_3^2}\left(\sum_{j=28}^{36} c_j S^6_j\right)(\boldsymbol{x}) = \frac{\partial^2}{\partial \beta_3^2}\left(c_{28} S^6_{28} + c_{29} S^6_{29} + c_{30} S^6_{30}\right)(\boldsymbol{x}) \\ &= 120 * 4 c_{28} \beta_1^3 \beta_2 + 270 * 4 c_{29} \beta_1^2 \beta_2^2 + 120 * 4 c_{30} \beta_1 \beta_2^3, \end{aligned}$$

a linear combination of linearly independent univariate Bernstein polynomials of degree 4. It follows that $c_{28} = c_{29} = c_{30} = 0$. Moving around this ring we conclude that $c_j = 0$, $j \leq 33$. Finally, by taking third derivatives with respect to β_3 we obtain by (1.46)

$$0 = \frac{\partial^3}{\partial \beta_3^3}\left(\sum_{j=34}^{36} c_j S^6_j\right)(\boldsymbol{x}) = \frac{\partial^3}{\partial \beta_3^3}\left(c_{34} S^6_{34} + c_{35} S^6_{35}\right)(\boldsymbol{x}) = 360 * 6\left(\beta_1^2 \beta_2 c_{34} + \beta_1 \beta_2^2 c_{35}\right).$$

This implies that $c_{34} = c_{35} = 0$ and then $c_{36} = 0$ since by (1.46) $S^6_{36}|_{\mathcal{T}_3}$ is nonzero.
■

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